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On the local Borel transform of Perturbation Theory

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Abstract

We prove existence of the local Borel transform for the perturbative series of massive φ_4^4-theory. As compared to previous proofs in the literature, the present bounds are much sharper as regards the dependence on external momenta, they are explicit in the number of external legs, and they are obtained quite simply through a judiciously chosen induction hypothesis applied to the Wegner-Wilson-Polchinski flow equations. We pay attention not to generate an astronomically large numerical constant for the inverse radius of convergence of the Borel transform.

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1 Introduction

Perturbation theory in quantum field theory is suspected to be divergent. The divergent behaviour can be directly related to the presence of nontrivial minima of the classical action in the complex coupling constant plane [Li], and one speaks of instanton singularities in consequence. Starting from the expansion in terms of Feynman diagrams the singularity can also be related to the increase of the number of Feynman diagrams at high orders in perturbation theory. In theories like φ^4, this number grows as $N!$, where N is the order of perturbation theory. This indicates divergent behaviour. In four dimensions this divergence has never been proven however. The main obstruction stems from the renormalization subtractions which are required to cancel short distance singularities. They lead to the appearance of contributions of opposite sign in the Feynman amplitudes. A lower bound on perturbative contributions would then require to control the absence of efficient sign cancellations, a task which has turned out to be too difficult up to the present day. Thus divergence can only be proven in three or fewer dimensions where the renormalization problem is marginal or absent [Sp], [Br], [MR]. In the four-dimensional case the very need for renormalization implies the appearance of a new (hypothetical) source of divergence of the perturbative expansion, named renormalon singularity after 't Hooft [tH]. This type of singularity is related - in the language of Feynman graphs - to the presence of graphs which require a number of renormalization subtractions proportional to the order of perturbation theory. In a strictly renormalizable theory it typically leads to a corresponding power of the logarithms of the momenta flowing through the diagram. For example for the diagram of Fig.1 we obtain an integral of the type

$$\int d^4p \, \frac{1}{(p^2+m^2)^3} \, \log^N(\frac{p^2+m^2}{m^2}) \sim N! \, ,$$

where N is the number of bubble graph insertions and p the momentum flowing through the big loop. Such a behaviour is obviously not compatible with a convergent perturbation expansion.

It was then proven in the seminal work of de Calan and Rivasseau [CR] that the two sources of divergent behaviour do not conspire to deteriorate the situation even more. Even in the presence of both instanton and renormalon type singularities the Borel transform of the perturbation expansion has a finite radius of convergence, i.e. perturbative amplitudes at order N do not grow more rapidly than $N!$. In fact one of the main results of [CR] is that the number of graphs which require $k \leq N$ renormalization subtractions is bounded by $(const)^N \, \frac{N!}{k!}$ so that the bound they present on their amplitudes, which is of the form $(const')^N \, k!$, is sufficient to prove local existence of the Borel transform.

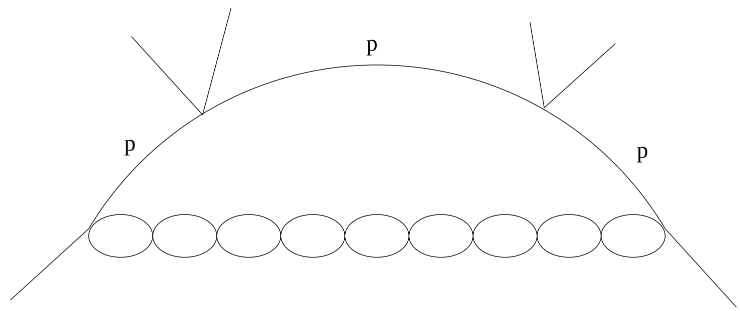

Figure 1: A renormalon diagram in φ_4^4-theory

The subject of large orders of perturbation theory was taken up by several authors in the sequel. The bounds were improved and generalized in the paper [FMRS]. In [CPR] the result was extended to massless φ_4^4-theory. Local existence of the Borel transform for QED was proven in the book [FHRW]. David, Feldman and Rivasseau [DFR] made essential progress in proving that the radius of convergence of the Borel transformed series for the φ_4^4-theory is not smaller than what is expected from the analysis of typical simple graphs contributing to the renormalon singularity as the one of Fig.1. Namely they showed that this radius is bounded below by the inverse of the first coefficient of the β-function, as suspected by 't Hooft. In fact this coefficient is calculated from a subclass of diagrams of which the one shown in Fig.1 is a representative. They are obtained by iteratively replacing in all possible ways elementary vertices by the one-loop bubble graph which apppears as a multiple insertion in Fig.1. The proof required a judicious partial resummation technique applied to the perturbative expansion, of a similar kind as the one employed previously in [Ri] to prove the existence (beyond perturbation theory) of planar "wrong" sign φ_4^4-theory. It also made use of the precise upper bounds on the perturbative series in the absence of renormalon type diagrams established previously in [MR] and [MNRS]. Finally Keller [Ke] first proved the local existence of the Borel transform in the framweork of the Wegner-Wilson-Polchinski flow equations which we also use in this paper.

As compared to the previous papers our motivation and in consequence the results are different. Our paper is of course closest in spirit to [Ke], which is the only one where the dependence on the number of external legs is explicitly controlled. The paper is part of a larger program to get rigorous control of the properties of the Schwinger or Green functions of quantum field theory with the aid of flow equations. A review is in [Mü], for recent novel results see e.g. [KMü], [Ko]. Our aim is not only to control the large order behaviour

of perturbation theory in the sense of the mathematical statement on the existence of the local Borel transform. We would like to control the whole set of Schwinger functions at the same time as regards their large momentum behaviour. This is in fact necessary if the bounds on the Schwinger functions are supposed to serve as an ingredient to further analysis. If for example they appear as an input in the flow equations, or simliarly in Schwinger-Dyson type equations, bad bounds on one side will typically undermine good ones on the other side ; for example bad high momentum behaviour will lead to bad high order behaviour when closing loops and integrating over loop momenta. In the same way, since an n-point function can be otained by merging two external lines and forming a loop in an $(n+2)$-point function, bounds which are not sufficiently strong as regards the dependence on n, will not be of much use either. We need bounds on the high momentum behaviour which do not increase faster than logarithmically with momentum (apart from the two-pont function), and which are thus optimal for the four-point function, in the sense that they are saturated by certain individual Feynman amplitudes. Such bounds were proven in [KM], however without control on the behaviour at large orders of perturbation theory or at large number of external legs. In the above cited papers the control on the high momentum behaviour is far from sufficient, in [CR] and in [Ke] the radius of convergence of the Borel transform shrinks as an inverse power of momentum, in the other papers the result is not framed in momentum space but rather in distributional sense making use of various norms, and certainly too far from optimal to be used in the above described context. We note that bounds in position space, if optimal in the above sense, could serve as well as those in momentum space. We adressed the problem in momentum space here since it is of more common use in short distance physics. For work with flow equations in position space see [KMü].

We would also like to stress the fact that we pay much attention to the fact not to produce astronomical[1] constants in the lower bounds on the inverse radius of convergence of the Borel transformed Schwinger functions. The paper could have been considerably shortened without that effort, and the reader will easily find his shortened path through the paper, if he is not interested in that aspect. The constants obtained in the literature are typically astronomically large ; in some restricted sense this is even true for the optimal result [DFR], since the bound obtained is on asymptotically large orders of perturbation theory, allowing smaller orders to be very large. In a closed system of equations it is again not possible to relax on low orders of perturbation theory without having a drawback on higher orders. Further considerable effort seems necessary if one wants to obtain a close

[1] an astronomical constant would be one of the form 10^n where n is a large integer. Our aim is to show that a small value of n can be accommodated for.

to realistic value for this inverse radius. It requires more explicit calculations in lowest orders which are of course doable since the flow equations provide an explicit calculational scheme.

Our paper is organized as follows. We first present the flow equation framework as we will use it in the proof. Then we collect some elementary auxiliary bounds which are to be used in the proof of the subsequent proposition. This part could be considerably shortened, were it not for the above mentioned aim to avoid the appearance of astronomical constants. Then we present our results and their proof. The reader familiar with the domain will realize that the proof is comparatively short and (hopefully) transparent. The hardest part of the work consisted in finding out the pertinent induction hypothesis.

2 The flow equation framework

Renormalization theory based on the flow equation (FE) [WH] of the renormalization group [Wi] has been exposed quite often in the literature [Po], [KKS], [Mü]. So we will introduce it rather shortly. The object studied is the regularized generating functional L^{Λ,Λ_0} of connected (free propagator) amputated Green functions (CAG). The upper indices Λ and Λ_0 enter through the regularized propagator

$$C^{\Lambda,\Lambda_0}(p) \;=\; \frac{1}{p^2+m^2}\{e^{-\frac{p^2+m^2}{\Lambda_0^2}} - e^{-\frac{p^2+m^2}{\Lambda^2}}\}$$

or its Fourier transform

$$\hat{C}^{\Lambda,\Lambda_0}(x) = \int_p C^{\Lambda,\Lambda_0}(p)\, e^{ipx} \;, \qquad \text{with} \quad \int_p := \int_{\mathbb{R}^4} \frac{d^4p}{(2\pi)^4} \;. \tag{1}$$

We assume $0 \le \Lambda \le \Lambda_0 \le \infty$ so that the Wilson flow parameter Λ takes the role of an infrared (IR) cutoff[2], whereas Λ_0 is the ultraviolet (UV) regularization. The full propagator is recovered for $\Lambda = 0$ and $\Lambda_0 \to \infty$. For the "fields" and their Fourier transforms we write $\hat{\varphi}(x) = \int_p \varphi(p)\; e^{ipx}$, $\frac{\delta}{\delta\hat{\varphi}(x)} = (2\pi)^4 \int_p \frac{\delta}{\delta\varphi(p)}\, e^{-ipx}$. For our purposes the fields $\hat{\varphi}(x)$ may be assumed to live in the Schwartz space $\mathcal{S}(\mathbb{R}^4)$. For finite Λ_0 and in finite volume the theory can be given rigorous meaning starting from the functional integral

$$e^{-\frac{1}{\hbar}(L^{\Lambda,\Lambda_0}(\hat{\varphi})+I^{\Lambda,\Lambda_0})} \;=\; \int d\mu_{\Lambda,\Lambda_0}(\hat{\phi})\; e^{-\frac{1}{\hbar}L_0(\hat{\phi}+\hat{\varphi})} \;. \tag{2}$$

[2]Such a cutoff is of course not necessary in a massive theory. The IR behaviour is only modified for Λ above m.

On the r.h.s. of (2) $d\mu_{\Lambda,\Lambda_0}(\hat{\phi})$ denotes the (translation invariant) Gaussian measure with covariance $\hbar\hat{C}^{\Lambda,\Lambda_0}(x)$. The functional $L_0(\hat{\varphi})$ is the bare action including counterterms, viewed as a formal power series in $\hbar$. Its general form for symmetric φ_4^4 theory is

$$L^{\Lambda_0,\Lambda_0}(\hat{\varphi}) = \frac{g}{4!}\int d^4x\, \hat{\varphi}^4(x) +$$

$$+ \int d^4x\, \{\frac{1}{2}\, a(\Lambda_0)\ \hat{\varphi}^2(x) + \frac{1}{2}\, b(\Lambda_0)\ \sum_{\mu=0}^{3}(\partial_\mu\hat{\varphi})^2(x) + \frac{1}{4!}\, c(\Lambda_0)\ \hat{\varphi}^4(x)\}\ , \tag{3}$$

the parameters $a(\Lambda_0),\ b(\Lambda_0),\ c(\Lambda_0)$ fulfill

$$a(\Lambda_0),\ c(\Lambda_0) = O(\hbar)\,, \quad b(\Lambda_0) = O(\hbar^2)\ . \tag{4}$$

They are directly related to the standard mass, coupling constant and wave function counterterms. On the l.h.s. of (2) there appears the normalization factor $e^{-I^{\Lambda,\Lambda_0}}$ which is due to vacuum contributions. The exponent I^{Λ,Λ_0} diverges in infinite volume so that we can take the infinite volume limit only when it does not appear any more. We do not make the finite volume explicit here since it plays no role in the sequel. For a more thorough discussion see [Mü], [KMR].

The FE is obtained from (2) on differentiating w.r.t. Λ. It is a differential equation for the functional L^{Λ,Λ_0} :

$$\partial_\Lambda(L^{\Lambda,\Lambda_0} + I^{\Lambda,\Lambda_0}) = \tag{5}$$
$$= \frac{\hbar}{2}\,\langle\frac{\delta}{\delta\hat{\varphi}},(\partial_\Lambda\hat{C}^{\Lambda,\Lambda_0})\frac{\delta}{\delta\hat{\varphi}}\rangle L^{\Lambda,\Lambda_0} - \frac{1}{2}\,\langle\frac{\delta}{\delta\hat{\varphi}}L^{\Lambda,\Lambda_0},(\partial_\Lambda\hat{C}^{\Lambda,\Lambda_0})\frac{\delta}{\delta\hat{\varphi}}L^{\Lambda,\Lambda_0}\rangle\ .$$

By $\langle\ ,\ \rangle$ we denote the standard scalar product in $L^2(\mathbb{R}^4, d^4x)$. Changing to momentum space and expanding in a formal powers series w.r.t. $\hbar$ we write

$$L^{\Lambda,\Lambda_0}(\varphi)\ =\ \sum_{l=0}^{\infty}\hbar^l\, L_l^{\Lambda,\Lambda_0}(\varphi)\,.$$

From $L_l^{\Lambda,\Lambda_0}(\varphi)$ we then define the CAG of order l in momentum space through

$$\delta^{(4)}(p_1+\ldots+p_n)\,\mathcal{L}_{n,l}^{\Lambda,\Lambda_0}(p_1,\ldots,p_{n-1})\ =\ \frac{1}{n!}(2\pi)^{4(n-1)}\delta_{\varphi(p_1)}\ldots\delta_{\varphi(p_n)}L_l^{\Lambda,\Lambda_0}|_{\varphi\equiv 0}\ \ , \tag{6}$$

where we have written $\delta_{\varphi(p)}\ =\ \delta/\delta\varphi(p)$. The CAG are symmetric in their momentum arguments by definition. Note that by our definitions the free two-point function is not contained in $L_l^{\Lambda,\Lambda_0}(\varphi)$, since it is attributed to the Gaussian measure in (2). This is important for the set-up of the inductive scheme, from which we will prove our bounds below. We thus define

$$\mathcal{L}_{2n,l}^{\Lambda,\Lambda_0} \equiv 0\ \text{ for } l<0\,,\ n\geq 1\ , \quad \text{and} \quad \mathcal{L}_{2,0}^{\Lambda,\Lambda_0}\equiv 0\ .$$

The FE (5) rewritten in terms of the CAG (6) takes the following form

$$\partial_\Lambda \partial^w \mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}(p_1,\ldots p_{n-1}) = \binom{2n+2}{2} \int_k (\partial_\Lambda C^{\Lambda,\Lambda_0}(k))\, \partial^w \mathcal{L}^{\Lambda,\Lambda_0}_{2n+2,l-1}(k,-k,p_1,\ldots p_{2n-1}) \quad (7)$$

$$-\sum_{\substack{l_1+l_2=l,\\ w_1+w_2+w_3=w\\ n_1+n_2=n+1}} 2\, n_1\, n_2\, c_{\{w_j\}} \Big[\partial^{w_1} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_1,l_1}(p_1,\ldots,p_{2n_1-1})\ (\partial^{w_3}\partial_\Lambda C^{\Lambda,\Lambda_0}(q))\ \partial^{w_2} \mathcal{L}^{\Lambda,\Lambda_0}_{2n_2,l_2}(p_{2n_1},\ldots,p_{2n-1}) \Big]_{sy}$$

$$\text{with} \quad q = -p_1 - \ldots - p_{2n_1-1} \;=\; -p_{2n_1} \;=\; p_{2n_1+1} + \ldots + p_{2n}\ .$$

Here we have written (7) directly in a form where also momentum derivatives of the CAG (6) are performed. In this paper we will restrict for simplicity to up to 3 derivatives all taken w.r.t. *one* momentum p_i, since our aim is in the first place to bound the Schwinger functions themselves, and not their derivatives [3]. We use the shorthand[4] notations

$$\partial^w := \prod_{\mu=0}^{3} \Big(\frac{\partial}{\partial p_{i,\mu}}\Big)^{w^\mu} \quad \text{with} \quad w = (w^0,\ldots,w^3)\ , \quad |w| = \sum_\mu w^\mu$$

and

$$w! \;=\; w^0! \ldots w^3!\ , \quad c_{\{w_j\}} \;=\; \frac{w!}{w_1!\, w_2!\, w_3!}\ .$$

The symbol sy means taking the mean value over those permutations π of $(1,\ldots,2n)$, for which $\pi(1) < \pi(2) < \ldots < \pi(2n_1-1)$ and $\pi(2n_1) < \pi(2n_1+1) < \ldots < \pi(2n)$.
For the derivatives of the propagator we find the following relations

$$\partial_\Lambda C^{\Lambda,\Lambda_0}(p) = -\frac{2}{\Lambda^3}\, e^{-\frac{p^2+m^2}{\Lambda^2}}\ , \quad \partial_{p_\mu} e^{-\frac{p^2+m^2}{\Lambda^2}} = -\frac{2\,p_\mu}{\Lambda^2}\, e^{-\frac{p^2+m^2}{\Lambda^2}}\ , \tag{8}$$

$$\partial_{p_\mu}\partial_{p_\nu} e^{-\frac{p^2+m^2}{\Lambda^2}} = \Big[\frac{4}{\Lambda^4}\, p_\mu\, p_\nu \;-\; \frac{2}{\Lambda^2}\, \delta_{\mu\nu}\Big] e^{-\frac{p^2+m^2}{\Lambda^2}}\ , \tag{9}$$

$$\partial_{p_\mu}\partial_{p_\nu}\partial_{p_\rho} e^{-\frac{p^2+m^2}{\Lambda^2}} = \Big[-\frac{8}{\Lambda^6}\, p_\mu\, p_\nu\, p_\rho \;+\; \frac{4}{\Lambda^4}\big(\delta_{\mu\nu}p_\rho \;+\; \delta_{\mu\rho}p_\nu \;+\; \delta_{\nu\rho}p_\mu\big)\Big]\, e^{-\frac{p^2+m^2}{\Lambda^2}}\ . \tag{10}$$

[3] In distributing the derivatives over the three factors in the second term on the r.h.s. with the Leibniz rule, we have tacitly assumed that the momentum p_i appears among those from $\mathcal{L}^{\Lambda,\Lambda_0}_{2n_1,l_1}$. If this is not the case one has to parametrize $\mathcal{L}^{\Lambda,\Lambda_0}_{2n_1,l_1}$ in terms of (say) $(p_2,\ldots p_{2n_1})$ with $p_{2n_1} = -p_{2n_1+1} - \ldots - p_{2n}$, to introduce the p_i-dependence in $\mathcal{L}^{\Lambda,\Lambda_0}_{2n_1,l_1}$. For an extensive systematic treatment including the general situation where derivatives w.r.t. several external momenta are present, see [GK]. This situation, also considered in [KM], could be analysed here too at the prize of basically notational complication.

[4] slightly abusive, since the index i is suppressed in w

3 A collection of elementary bounds

The subsequent lemmata state a number bounds which we will make recurrent use of in the proof of our main result.

Lemma 1 : For $l \in \mathbb{N}_0$

a)

$$\sum_{\substack{0 \le l_1, l_2, \\ l_1 + l_2 = l}} \frac{1}{(l_1+1)^2\,(l_2+1)^2} \;\le\; \frac{5}{(l+1)^2}\,, \qquad \sum_{\substack{1 \le l_1, l_2, \\ l_1 + l_2 = l}} \frac{1}{(l_1+1)^2\,(l_2+1)^2} \;\le\; \frac{3}{(l+1)^2}\,, \tag{11}$$

b)

$$\sum_{\substack{1 \le n_1, n_2, \\ n_1 + n_2 = n+1}} \frac{1}{n_1^3\, n_2^3} \;\le\; \frac{4}{n^3}\,, \qquad \sum_{\substack{2 \le n_1, n_2, \\ n_1 + n_2 = n+1}} \frac{1}{n_1^3\, n_2^3} \;\le\; \frac{2}{n^3}\,. \tag{12}$$

Proof : a) The inequality can be verified explicitly for $l \le 5$. Assuming $l > 5$ we have

$$\sum_{\substack{0 \le l_1, l_2, \\ l_1 + l_2 = l}} \frac{1}{(l_1+1)^2\,(l_2+1)^2} \;=\; \frac{2}{(l+1)^2} \;+\; \sum_{k=1}^{l-1} \frac{1}{(k+1)^2\,(l-k+1)^2} \tag{13}$$

$$\le \frac{2}{(l+1)^2} \;+\; \int_0^l \frac{dx}{(x+1)^2(l-x+1)^2} \;=\; \frac{2}{(l+1)^2} \;+\; \int_1^{l+1} dx \Big(\frac{a+bx}{x^2} + \frac{c-bx}{(l+2-x)^2} \Big)\,,$$

where

$$a \;=\; \frac{1}{(l+2)^2}\,, \quad b \;=\; \frac{2}{(l+2)^3}\,, \quad c \;=\; \frac{3}{(l+2)^2}\,.$$

The integral equals then

$$\frac{1}{(l+2)^2} \Big(2[1 - \frac{1}{l+1}] \;+\; \frac{4}{l+2}\, \log(l+1) \Big) \;\le \frac{3}{(l+1)^2} \quad \text{for} \quad l > 5\,, \tag{14}$$

and the bound is thus also verified for $l > 5$. The second statement in (11) is a direct consequence of the first since a term $\frac{2}{(l+1)^2}$ is subtracted on the l.h.s.

b) We may again assume $n > 5$ on verifying the lowest values explicitly. The statement then follows from the proof of a) through

$$\sum_{\substack{1 \le n_1, n_2, \\ n_1 + n_2 = n+1}} \frac{1}{n_1^3\, n_2^3} \;=\; \sum_{\substack{0 \le n_1, n_2, \\ n_1 + n_2 = n-1}} \frac{1}{(n_1+1)^3\,(n_2+1)^3}$$

$$\le\; \frac{2}{n^3} \;+\; \sup_{1 \le n_1 \le n-1} \frac{1}{(n_1+1)\,(n-n_1)} \sum_{\substack{1 \le n_1, n_2, \\ n_1 + n_2 = n-1}} \frac{1}{(n_1+1)^2\,(n_2+1)^2}$$

$$\leq \frac{2}{n^3} + \frac{1}{2(n-1)} \sum_{1 \leq n_1 \leq n-2} \frac{1}{(n_1+1)^2 \, (n-n_1)^2} \; \leq \; \frac{2}{n^3} + \frac{1}{2(n-1)} \, \frac{3}{n^2} \; \leq \; \frac{4}{n^3} \; ,$$

where we used the bound (14) on (13) in the last but second inequality. The second inequality in b) then follows directly from the previous calculation. ∎

Lemma 2 :

a) For integers $n \geq 3\,,\ n_1\,,\, n_2 \geq 1\,,\ \ l\,,\, l_1\,,\, \lambda_1\,,\ l_2\,,\, \lambda_2 \geq 0$

$$\sum_{\substack{l_1+l_2=l,\\ n_1+n_2=n+1,\\ \lambda_1 \leq l_1,\, \lambda_2 \leq l_2,\\ \lambda_1+\lambda_2=\lambda}} \frac{1}{(l_1+1)^2\,(l_2+1)^2\, n_1^2\, n_2^2} \; \frac{n!}{n_1!\, n_2!} \; \frac{\lambda!}{\lambda_1!\, \lambda_2!} \; \frac{(n_1+l_1-1)!\; (n_2+l_2-1)!}{(n+l-1)!}$$

$$\leq \; K_0 \, \frac{1}{(l+1)^2} \, \frac{1}{n^2} \; , \qquad \text{where we may choose} \qquad K_0 = 20 \; . \tag{15}$$

For $n_1, n_2 \, \geq 2$

$$\sum_{\substack{l_1+l_2=l,\\ n_1+n_2=n+1,\\ \lambda_1 \leq l_1,\, \lambda_2 \leq l_2,\\ \lambda_1+\lambda_2=\lambda}} \frac{1}{(l_1+1)^2\,(l_2+1)^2\, n_1^2\, n_2^2} \; \frac{n!}{n_1!\, n_2!} \; \frac{\lambda!}{\lambda_1!\, \lambda_2!} \; \frac{(n_1+l_1-1)!\; (n_2+l_2-1)!}{(n+l-1)!}$$

$$\leq \; \frac{1}{2} \, K_0 \, \frac{1}{(l+1)^2} \, \frac{1}{n^2} \; . \tag{16}$$

b) For $n \geq 3\,,\ n_1 = 2\,,\ n_2 = n-1$

$$\sum_{\substack{l_1+l_2=l,\\ \lambda_1 \leq l_1,\, \lambda_2 \leq l_2,\\ \lambda_1+\lambda_2=\lambda}} \frac{1}{(l_1+1)^2\,(l_2+1)^2\, n_1^2\, n_2^2} \; \frac{n!}{n_1!\, n_2!} \; \frac{\lambda_1!}{\lambda_1!\, \lambda_2!} \; \frac{(n_1+l_1-1)!\; (n_2+l_2-1)!}{(n+l-1)!}$$

$$\leq \; K_0' \, \frac{1}{(l+1)^2} \, \frac{1}{n^2} \; , \qquad \text{where we may choose} \qquad K_0' = (\frac{3}{4})^3 \cdot 5 \; \leq \; 2.2 \; . \tag{17}$$

c) For $n \geq 2\,,\ n_1 = 1\,,\ n_2 = n$

$$\sum_{\substack{l_1+l_2=l,\\ \lambda_1 \leq l_1,\, \lambda_2 \leq l_2,\\ \lambda_1+\lambda_2=\lambda}} \frac{1}{(l_1+1)^2\,(l_2+1)^2\, n_1^2\, n_2^2} \; \frac{n!}{n_1!\, n_2!} \; \frac{\lambda!}{\lambda_1!\, \lambda_2!} \; \frac{(n_1+l_1-1)!\; (n_2+l_2-1)!}{(n+l-1)!}$$

$$\leq \; K_0'' \, \frac{1}{(l+1)^2} \, \frac{1}{n^2} \; , \qquad \text{where we may choose} \qquad K_0'' = 5 \; . \tag{18}$$

Proof : a) We have

$$\frac{n!}{n_1!\,n_2!}\,\frac{\lambda!}{\lambda_1!\,\lambda_2!}\,\frac{(n_1+l_1-1)!\,(n_2+l_2-1)!}{(n+l-1)!} \;=\; \frac{n}{n_1\,n_2}\binom{n-1}{n_1-1}\binom{\lambda}{\lambda_1}\left[\binom{n+l-1}{n_1+l_1-1}\right]^{-1} .$$

We note that

$$\binom{n-1}{n_1-1}\binom{l}{l_1} \;\le\; \binom{n+l-1}{n_1+l_1-1} . \tag{19}$$

This follows directly from the standard identity

$$\sum_{k=0}^{p}\binom{n-1}{p-k}\binom{l}{k} \;=\; \binom{n+l-1}{p} ,$$

assuming without limitation that $n-1 \ge l$ and setting $p = \inf\{n_1+l_1-1, n+l-(n_1+l_1)\} \le \frac{n+l-1}{2} \le n-1$.

Secondly we show that for $l = l_1 + l_2$

$$\sum_{\substack{\lambda_1\le l_1,\,\lambda_2\le l_2,\\ \lambda_1+\lambda_2=\lambda}} \frac{\lambda!}{\lambda_1!\,\lambda_2!} \;\le\; \binom{l}{l_1} . \tag{20}$$

For the inductive proof we assume $l \ge 1$ and without loss $l_2 \le l_1$. To realize by induction on $0 \le k \le l_2$ that

$$A_k \;:=\; \left[\binom{l}{l_1}\right]^{-1} \sum_{\substack{\lambda_1\le l_1,\,\lambda_2\le l_2,\\ \lambda_1+\lambda_2=l-k}} \frac{(l-k)!}{\lambda_1!\,\lambda_2!} \;\le 1 ,$$

we start from $A_0 = 1$. Then assuming that we have $A_{k-1} \le 1$ for $k \ge 1$ we find

$$A_k \;=\; \frac{l_1-(k-1)}{l-(k-1)}\,A_{k-1} + \left[\binom{l}{l_1}\right]^{-1}\binom{l-k}{l_1} \;\le\; 1 - \frac{l_2}{l-(k-1)} + \frac{l_2}{l}\,\frac{(l_2-1)\dots(l_2-(k-1))}{(l-1)\dots(l-(k-1))} .$$

This equals 1 for $k = 1$ and can be bounded for $k \ge 2$ through

$$1 - \frac{l_2}{l-(k-1)}\Bigl(1 - \frac{(l_2-1)(l_2-2)\dots(l_2-(k-1))}{l\quad (l-1)\;\dots\;(l-(k-2))}\Bigr) \;\le\; 1 .$$

For $l_2 < k \le l$ it is immediate to see that $A_k \le A_{k-1}$ since the sum for A_k does not contain more nonvanishing terms than the one for A_{k-1}, and a nonvanishing term in A_k can be bounded by a corresponding one in A_{k-1} :

$$\frac{(l-k)!}{\lambda_1!\,\lambda_2!} \;\le\; \frac{(l-(k-1))!}{(\lambda_1+1)!\,\lambda_2!} .$$

Now it follows from (19), (20) that

$$\sum_{\substack{\lambda_1 \le l_1,\, \lambda_2 \le l_2\,, \\ \lambda_1+\lambda_2=\lambda}} \frac{n}{n_1\, n_2}\, \frac{(n_1+l_1-1)!}{(n_1-1)!\,\lambda_1!}\, \frac{(n_2+l_2-1)!}{(n_2-1)!\,\lambda_2!}\, \frac{(n-1)!\,\lambda!}{(n-1+l)!} \;\le\; \frac{n}{n_1\, n_2}\,. \tag{21}$$

Using Lemma 1 we then get

$$\sum_{\substack{l_1+l_2=l, \\ n_1+n_2=n+1}} \frac{n}{n_1\, n_2}\, \frac{1}{(l_1+1)^2\,(l_2+1)^2\, n_1^2\, n_2^2} \;\le\; \frac{20}{(l+1)^2\, n^2}\,. \tag{22}$$

The statements (16) and parts b) (17) and c) (18) follow from Lemma 1 and (21). ■

Lemma 3 : For $v \le 3$ and $a_i\,,\, x \in \mathbb{R}^4$ the following inequality holds

$$e^{-\frac{x^2}{2}}\, \prod_{i=1}^{v} \frac{1}{\sup(1, |x+a_i|)} \;\le\; c(v)\, \prod_{i=1}^{v} \frac{1}{\sup(1, |a_i|)}\,, \tag{23}$$

where we may choose

$$c(0) \,=\, 1\,, \quad c(1) \,=\, 1.4\,, \quad c(2) \,=\, 2.5\,, \quad c(3) \,=\, 5.25\,. \tag{24}$$

Proof : The inequality is trivial if one allows for large constants. Suppose $v = 3$. We may suppose without limitation that $|a_3| \ge |a_2| \ge |a_1| \ge 1$ (if $a_i \le 1$ we may pass to the case $v-1$), and that $|x| \le \sup|a_i|$ since the expression on the l.h.s. of (23) is maximized if all $a_i \in \mathbb{R}^4$ are parallel and anti-parallel to x. In this case, assumming that $|a_3|\,|a_2|\,|a_1| \,\ge (1+|x|)^3$, the inequality at fixed product $|a_3|\,|a_2|\,|a_1|$ and at fixed $|x|$, becomes most stringent if $|a_1|$, $|a_2| = 1+|x|$. It then takes the form

$$e^{-\frac{x^2}{2}}\, (1+|x|)^2 \;\le\; c(3)\, \frac{|a_3|-|x|}{|a_3|} \quad \text{with} \quad |a_3| > 1+|x|\,. \tag{25}$$

If $|a_3|\,|a_2|\,|a_1| \,<\, (1+|x|)^3$, the bound is satisfied if we demand

$$e^{-\frac{x^2}{2}} \;\le\; c(3)\, \frac{1}{(1+|x|)^3}\,.$$

This relation is also sufficient for (25) to hold. The expression $e^{-\frac{x^2}{2}}\,(1+|x|)^3$ is maximal for $|x| = \frac{\sqrt{13}-1}{2}$ and bounded by 5.25. The cases $v = 2$ and $v = 1$ are treated analogously. ■

Lemma 4 : For $r \in \mathbb{N}$ and $a \geq 0$

$$\int_x e^{-\frac{|x|^2}{2}} \log^r(|x|+a) \ \leq \ \frac{1}{4} \log_+^r a \ + \ \frac{1}{3} (r!)^{1/2} \ , \tag{26}$$

where $\log_+ x \ := \log(\sup(1,x))$.

Proof : Again the only nontrivial point is to avoid bad numerical constants in the bound. Remembering the definition (1), first note that for $r \leq 6$, $a \leq 3$

$$\int_x e^{-\frac{|x|^2}{2}} \log^r(|x|+a) \ \leq \ \int_x e^{-\frac{|x|^2}{2}} \log^r 5 \ + \ \int_{|x|\geq 2} e^{-\frac{|x|^2}{2}} \log^r(|x|+a)$$

$$\leq \ \frac{(1.61)^r}{4\pi^2} \ + \ \frac{1}{8\pi^2} \sum_{n\geq 2} e^{-n^2/2}\, n^3 \ \log^r(3+n) \ \leq \ \frac{1}{3} \ (r!)^{1/2} \quad \surd \tag{27}$$

on bounding the sum numerically ; we also used the fact that the derivative of the integrand w.r.t. $|x|$ is negative for $|x| \geq 2$. Secondly, for $r \leq 6$, $a > 3$

$$(\log a)^{-r} \int_x e^{-\frac{|x|^2}{2}} \log^r(|x|+a) \ = \ \int_x e^{-\frac{|x|^2}{2}} \ \big[1 \ + \ \frac{\log(1+\frac{|x|}{a})}{\log a}\big]^r \tag{28}$$

$$\leq \ \int_x e^{-\frac{|x|^2}{2}} \ (1+\frac{\log 2}{\log a})^r \ + \ \frac{1}{8\pi^2} \sum_{n\geq 3} e^{-n^2/2}\, n^3 \big[1 \ + \ \frac{\log(1+\frac{n}{3})}{\log 3}\big]^6 \ \leq \ \frac{1}{4\pi^2} \ (1+\frac{\log 2}{\log a})^r \ + \ \frac{6.5}{8\pi^2}$$

$$\leq \ \frac{1}{4} \ + \ \frac{1}{3} \, \frac{(r!)^{1/2}}{\log^r a} \quad \surd$$

on bounding the sum numerically and on noting that the last inequality is valid taking $a = 3$ on the l.h.s. and $a = 5$ on the r.h.s., and also for $a = 5$ on the l.h.s. and $a = e^2$ on the r.h.s. For $\log a \geq 2$ the last bound can be replaced by $\frac{1}{4}$ independently of $r \leq 6$. Thirdly, for $r > 6$, $a \leq r$

$$\int_x e^{-\frac{|x|^2}{2}} \log^r(|x|+a) \ \leq \ \int_x e^{-\frac{|x|^2}{2}} \log^r(|x|+r) \ \leq \ \log^r r \int_x e^{-\frac{|x|^2}{2}} \ \big[1 \ + \ \frac{|x|}{r \, \log r}\big]^r$$

$$\leq \ \log^r r \int_x e^{-\frac{|x|^2}{2}+\frac{|x|}{\log r}} \ \leq \ \log^r r \ \frac{e^{\frac{1}{2\log^2 6}}}{4\pi^2} \int_{-\frac{1}{\log 6}}^{\infty} e^{-z} \ z \ dz \ \leq \ \frac{1}{10} \ \log^r r$$

on majorizing for $r = 6$ and completing the square in the last but second integral. Then

$$\frac{1}{10} \ \log^r r \leq \frac{1}{3} \ (r!)^{1/2} \ ,$$

noting that $\log^r r/(r!)^{1/2} \leq 2.75$, the maximal value being attained for $r = 15$.
In the fourth place we have for $a > r > 6$ quite similarly

$$\frac{1}{\log^r a} \int_x e^{-\frac{|x|^2}{2}} \log^r(|x|+a) \ = \int_x e^{-\frac{|x|^2}{2}} \ \big[1+\frac{\log(1+\frac{|x|}{a})}{\log a}\big]^r \ \leq \int_x e^{-\frac{|x|^2}{2}} \ \big[1+\frac{|x|}{r \, \log r}\big]^r \ \leq \ \frac{1}{10} \, .$$

$\blacksquare$

Lemma 5 : For $s \in \mathbb{N}\,,\ a > 0\,,\ M > \kappa \geq m > 0$

$$\sum_{\lambda=0}^{\lambda=l} \frac{1}{2^\lambda\,\lambda!} \int_\kappa^M d\kappa'\ \kappa'^{-s-1}\ \log^\lambda(\sup(\frac{a}{\kappa'},\frac{\kappa'}{m}))\ \leq\ 3\ \frac{\kappa^{-s}}{s} \sum_{\lambda=0}^{\lambda=l} \frac{1}{2^\lambda\,\lambda!}\ \log^\lambda \sup(\frac{a}{\kappa},\frac{\kappa}{m})\ . \tag{29}$$

Proof : We have

$$\int_\kappa^M d\kappa'\ \kappa'^{-s-1}\ \log^\lambda(\sup(\frac{a}{\kappa'},\frac{\kappa'}{m}))\ \leq \frac{\kappa^{-s}}{s}\ \log_+^\lambda(\frac{a}{\kappa}) + \int_{\sup(\kappa,\sqrt{ma}\,)}^{\sup(\sqrt{ma},\,M)} d\kappa'\ \kappa'^{-s-1}\ \log^\lambda(\frac{\kappa'}{m})\ ,$$

and the last integral can be bounded by

$$\int_\kappa^M d\kappa'\ \kappa'^{-s-1}\ \log^\lambda(\frac{\kappa'}{m})\ \leq\ \frac{\kappa^{-s}}{s}\ \lambda! \sum_{\nu=0}^{\lambda} \frac{\log^\nu(\frac{\kappa}{m})}{\nu!}\ \frac{1}{s^{\lambda-\nu}}\ . \tag{30}$$

We then find

$$\sum_{\lambda=0}^{\lambda=l} \Big\{ \frac{1}{2^\lambda\,\lambda!}\ \log_+^\lambda(\frac{a}{\kappa})\ +\ \frac{1}{2^\lambda}\ \sum_{\nu=0}^{\lambda} \frac{\log^\nu(\frac{\kappa}{m})}{\nu!}\ \frac{1}{s^{\lambda-\nu}} \Big\}$$

$$\leq\ \sum_{\lambda=0}^{\lambda=l} \frac{\log^\lambda(\frac{a}{\kappa})}{2^\lambda\,\lambda!}\ +\ 2 \sum_{\lambda=0}^{\lambda=l} \frac{\log^\lambda(\frac{\kappa}{m})}{2^\lambda\,\lambda!}\ \leq\ 3 \sum_{\lambda=0}^{\lambda=l} \frac{1}{2^\lambda\,\lambda!}\ \log^\lambda \sup(\frac{a}{\kappa},\frac{\kappa}{m})\ .$$

$\blacksquare$

Lemma 6 :

Here and in the following we set $\kappa =\ \Lambda + m\,$.

a)

$$\frac{2}{\Lambda^3}\ e^{-\frac{m^2}{\Lambda^2}}\ \leq\ \frac{K_2}{\kappa^3}\ , \qquad \text{where}\ \ K_2 = 6.2\ , \tag{31}$$

$$p^2\ e^{-\frac{p^2}{2\Lambda^2}}\ \leq\ \kappa^2\ \frac{2}{e}\ , \qquad |p|\ e^{-\frac{p^2}{2\Lambda^2}}\ \leq\ \kappa\ \frac{1}{\sqrt{e}}\ . \tag{32}$$

b) For $|w| \leq 3$:

$$|\partial^w\, \frac{2}{\Lambda^3}\, e^{-\frac{p^2}{\Lambda^2}}\, e^{-\frac{m^2}{\Lambda^2}}|\ \leq\ K^{(|w|)}\ \kappa^{-3}\ [\sup(\kappa,|p|)]^{-|w|}\ . \tag{33}$$

with $\quad K^{(0)} = K_2\,,\ \ K^{(1)} = \frac{2K_2}{e}\ =\ 4.6\,,\ \ K^{(2)} = 77.5\,,\ \ K^{(3)} = 37\,.$

$$|\partial^w\, \frac{2}{\Lambda^3}\, e^{-\frac{p^2}{2\Lambda^2}}\, e^{-\frac{m^2}{\Lambda^2}}|\ \leq\ K'^{(|w|)}\ \kappa^{-3}\ [\sup(\kappa,|p|)]^{-|w|}\ . \tag{34}$$

$$\text{with} \quad K'^{(0)} = K_2\,, \;\; K'^{(1)} = \tfrac{4K_2}{e} \;=\; 9.2\,, \;\; K'^{(2)} = 135\,, \;\; K'^{(3)} = 407\,.$$

c) For $0 \le \tau \le 1$ and $p_4(\tau) = -\tau p_1 - p_2 - p_3$:

$$\frac{|p_1|\; e^{-\frac{p_1^2}{2\Lambda^2}}}{\sup(\kappa, \eta^{(4)}_{1,4}(\tau p_1, p_2, p_3, p_4(\tau)))} \;\le\; e^{-1/2}\;, \quad \frac{|p_1^2|\; e^{-\frac{p_1^2}{2\Lambda^2}}}{\sup(\kappa, \eta^{(4)}_{1,4}(\tau p_1, p_2, p_3, p_4(\tau)))^2} \;\le\; \frac{2}{e}\;, \tag{35}$$

where η is defined below (43),

$$\frac{|p|\; e^{-\frac{p^2}{2\Lambda^2}}}{\sup(\tau|p|,\kappa)} \;\le\; \frac{1}{\sqrt{e}}\;, \quad \frac{p^2\; e^{-\frac{p^2}{2\Lambda^2}}}{\sup(\tau|p|,\kappa)} \;\le\; \kappa\;\frac{2}{e} \quad , \quad \frac{|p|^3\; e^{-\frac{p^2}{2\Lambda^2}}}{\sup(\tau|p|,\kappa)} \;\le\; \kappa^2\; (\frac{3}{e})^{3/2}\;. \tag{36}$$

Proof : a) The bound (31) follows from

$$\frac{2}{\Lambda^3}\; e^{-\frac{m^2}{\Lambda^2}} \;\le\; \frac{2}{\kappa^3}\; \sup_{x\ge 0}(1+x)^3\; e^{-x^2}\;, \tag{37}$$

and the function of x is maximized for $x = \frac{\sqrt{7}-1}{2}$. To prove (32) note

$$p^2\; e^{-\frac{p^2}{2\Lambda^2}} \;\le\; \kappa^2\; \sup_x \Big\{ x^2\; e^{-\frac{x^2}{2}} \Big\} \;=\; \frac{2\kappa^2}{e}\;, \quad |p|\; e^{-\frac{p^2}{2\Lambda^2}} \;\le\; \kappa\; \sup_{x\ge 0} \Big\{ x\; e^{-\frac{x^2}{2}} \Big\} \;=\; \frac{\kappa}{\sqrt{e}}\;.$$

b) The bounds are proven similarly as in a). For $w = 0$ the result follows from a).
For $|w| = 1\,,2\,,3$ we use (8), (9),(10). We may suppose that the axes have been chosen such that p is parallel to one of them. For $|w| = 1$ we then find

$$|\kappa^3\; \partial^w\, \frac{2}{\Lambda^3}\, e^{-\frac{p^2}{2\Lambda^2}}\, e^{-\frac{m^2}{\Lambda^2}}|$$

$$\le \inf\Big\{ \frac{4}{|p|}\; \sup_{x^2}\{x^2\, e^{-\frac{x^2}{2}}\}\; \sup_{y\ge 0}\{(1+y)^3\, e^{-y^2}\}\,,\; \frac{4}{\kappa}\; \sup_{x\ge 0}\{x\, e^{-\frac{x^2}{2}}\}\; \sup_{y\ge 0}\{(1+y)^4\, e^{-y^2}\} \Big\}\;.$$

For $|w| = 2$ we obtain

$$|\kappa^3\; \partial^w\, \frac{2}{\Lambda^3}\, e^{-\frac{p^2}{2\Lambda^2}}\, e^{-\frac{m^2}{\Lambda^2}}|$$

$$\le\; \inf\Big\{ \frac{16}{|p|^2}\; \sup_{x^2}\{|x^4 - \frac{1}{2}x^2|\, e^{-\frac{x^2}{2}}\} \sup_{y\ge 0}\{(1+y)^3\, e^{-y^2}\,,\; \frac{16}{\kappa^2}\; \sup_{x\ge 0}\{|x^2 - \frac{1}{2}|\, e^{-\frac{x^2}{2}}\; \sup_{y\ge 0}\{(1+y)^5\, e^{-y^2}\} \Big\}\,.$$

For $|w| = 3$ we get

$$|\kappa^3\; \partial^w\, \frac{2}{\Lambda^3}\, e^{-\frac{p^2}{2\Lambda^2}}\, e^{-\frac{m^2}{\Lambda^2}}|$$

$$\le\; \inf\Big\{ \frac{16}{|p|^3}\; \sup_{x^2}\{|-x^6 + \frac{3}{2}x^4|\, e^{-\frac{x^2}{2}}\} \sup_{y\ge 0}\{(1+y)^3\, e^{-y^2}\},$$

$$\frac{16}{\kappa^3}\; \sup_{x\ge 0}\{|-x^3 + \frac{3}{2}x|\, e^{-\frac{x^2}{2}}\}\; \sup_{y\ge 0}\{(1+y)^6\, e^{-y^2}\} \Big\}\;.$$

Maximizing the expressions depending on x and y and taking the maximal constant in each of the three expressions gives the numerical constants of (34).
The bounds (33) follow on replacing $e^{-\frac{x^2}{2}} \to e^{-x^2}$ in maximizing the previous expressions.
c) The first bound (35) follows from

$$\frac{1}{\sup(\kappa, \eta^{(4)}_{1,4}(\tau p_1, p_2, p_3, p_4(\tau)))}\ |p_1|\ e^{-\frac{p_1^2}{2\Lambda^2}} \ \leq\ \frac{|p_1|}{\kappa}\ e^{-\frac{p_1^2}{2\Lambda^2}} \ \leq\ \frac{|p_1|}{\Lambda}\ e^{-\frac{p_1^2}{2\Lambda^2}} \ \leq\ e^{-1/2}$$

and the second bound follows analogously.
The bounds (36) are obtained by the same reasoning. ■

Lemma 7:
a)

$$\int_0^\Lambda d\Lambda'\ \Lambda'^{-3}\ e^{-m^2/\Lambda'^2}\ \kappa'^2\ \log^\lambda(\frac{\kappa'}{m})\ \leq\ K_1\ \frac{\log^{\lambda+1}(\frac{\kappa}{m})}{\lambda+1} \quad \text{with} \quad K_1\ =\ \frac{K_2}{2}\ =\ 3.1\,, \tag{38}$$

b)

$$\int_0^\Lambda d\Lambda'\ \Lambda'^{-5}\ e^{-m^2/\Lambda'^2}\ \kappa'^4\ \log^\lambda(\frac{\kappa'}{m})\ \leq\ K_1'\ \frac{\log^{\lambda+1}(\frac{\kappa}{m})}{\lambda+1} \quad \text{with} \quad K_1'\ =\ 14.5\ . \tag{39}$$

Proof: The integrals are bounded through

$$\int_1^{\kappa/m} \frac{dx}{x}\ (\frac{x}{x-1})^s\ e^{-\frac{1}{(x-1)^2}}\ \log^\lambda x\ \leq\ \sup_{y\geq 0}\Big((1+y)^s\ e^{-y^2}\Big)\frac{\log^{\lambda+1}(\kappa/m)}{\lambda+1}\ ,$$

where $s \in \{3,\,5\}$. The sup leads to the numerical constants. ■

Lemma 8: For $\lambda \in [0,1]$ and $x,\,y \in \mathbb{R}^d$, if $|x+y| \geq |x|$ then $|\lambda x + y| \geq \lambda|x|$.
Proof: $|\lambda x + y| \geq |x+y| - |(1-\lambda)x| \geq |x| - (1-\lambda)|x| = \lambda|x|$. ■

4 Sharp bounds on Schwinger functions

With the aid of the FE (7) it is possible to establish a particularly simple inductive proof of the renormalizability of φ_4^4 theory. Renormalizability in fact appears as a consequence of the following bounds [KKS], [Mü] on the functions $\mathcal{L}^{\Lambda,\Lambda_0}_{2n.l}$:

$$\text{Boundedness} \qquad |\partial^w \mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}(\vec{p})| \leq\ \kappa^{4-2n-|w|}\, \mathcal{P}_1(\log\frac{\kappa}{m})\, \mathcal{P}_2(\frac{|\vec{p}|}{\kappa})\,, \tag{40}$$

$$\text{Convergence} \quad |\partial_{\Lambda_0}\partial^w \mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}(\vec{p})| \le \frac{1}{\Lambda_0^2}\,\kappa^{5-2n-|w|}\,\mathcal{P}_3(\log\frac{\Lambda_0}{m})\,\mathcal{P}_4(\frac{|\vec{p}|}{\kappa})\ . \tag{41}$$

The $\mathcal{P}_i$ denote polynomials with nonnegative coefficients, which depend on $l, n, |w|$, but not on $\vec{p}$, Λ, $\kappa = \Lambda + m$, Λ_0. The statement (41) implies renormalizability, since it proves the limits $\lim_{\Lambda_0\to\infty,\,\Lambda\to 0}\mathcal{L}^{\Lambda,\Lambda_0}(\vec{p})$ to exist to all loop orders l. But the statement (40) has to be obtained first to prove (41).

The standard inductive scheme which is used to prove these bounds, and which we will also employ in the proof of the subsequent proposition, goes up in $n+l$ and for given $n+l$ descends in n, and for given n, l descends in $|w|$. The r.h.s. of the FE is then prior the l.h.s. in the inductive order, and the bounds can thus be verified for suitable boundary conditions on integrating the r.h.s. of the FE over Λ, using the bounds of the proposition. Terms with $2n + |w| \ge 5$ are integrated downwards from Λ_0 to Λ, since for those terms we have the boundary conditions at $\Lambda = \Lambda_0$ following from (3)

$$\partial^w\,\mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}(p_1,\ldots p_{2n-1}) = 0 \quad \text{for} \quad 2n + |w| \ge 5\,,$$

whereas the terms with $2n + |w| \le 4$ at the renormalization point - which we choose at zero momentum for simplicity - are integrated upwards from 0 to Λ, since they are fixed at $\Lambda = 0$ by renormalization conditions, which define the relevant parameters of the theory. We will choose for simplicity

$$\mathcal{L}^{0,\Lambda_0}_{4,l}(0,0,0) = \delta_{l,0}\,\frac{g}{4!}\ , \quad \mathcal{L}^{0,\Lambda_0}_{2,l}(0) = 0\ , \quad \partial_{p^2}\mathcal{L}^{0,\Lambda_0}_{2,l}(0) = 0\ , \tag{42}$$

though more general choices could be accommodated for without any problems[5].

Our new result combines the sharp bounds on the high momentum behaviour from [KM] with good control on the constants hidden in the symbols $\mathcal{P}$ in (40), (41) .

In the Theorem and the Proposition we use the following notations and assumptions : We denote by $(p_1,\ldots,p_{2n})$ a set of external momenta with $p_1+\ldots+p_{2n} = 0$, and we define

$$\vec{p} = (p_1,\ldots,p_{2n-1})\ , \qquad |\vec{p}| = \sup_{1\le i\le 2n}|p_i|\ .$$

Furthermore

$$\eta^{(2n)}_{i,j}(p_1,\ldots,p_{2n})\ :=\ \inf\Big\{|p_i + \sum_{k\in J} p_k|\ /\ J \subset \big(\{1,...,2n\} - \{i,j\}\big)\Big\}\ . \tag{43}$$

Thus $\eta^{(2n)}_{i,j}$ is the modulus of the smallest subsum of external momenta containing p_i but not p_j. We assume $0 \le \Lambda \le \Lambda_0$, and we write $\kappa = \Lambda + m$.

[5]It would amount to absorb the new constants in the respective lower bounds on K in part B of the proof.

Our main result can then be stated as follows :

Theorem :
There exists a constant $\tilde{K} > 0$ such that

$$|\mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}(\vec{p})| \le \kappa^{4-2n}\, \frac{\tilde{K}^{2l+n-2}}{n!}\, (n+l)! \sum_{\lambda=0}^{\lambda=l} \frac{\log^\lambda\big(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\big)}{2^\lambda\, \lambda!} \quad \text{for} \quad 2n > 2\,, \tag{44}$$

$$|\mathcal{L}^{\Lambda,\Lambda_0}_{2,l}(p)| \le \sup(|p|,\kappa)^2\, \frac{\tilde{K}^{2l}}{(l+1)^2}\, l! \sum_{\lambda=0}^{\lambda=l-1} \frac{\log^\lambda\big(\sup(\frac{|p|}{\kappa}, \frac{\kappa}{m})\big)}{2^\lambda\, \lambda!}\,, \quad l \ge 1\,. \tag{45}$$

The Theorem follows from the subsequent Proposition. In the Proposition the bounds are presented in a form such that they can serve at the same time as an induction hypothesis for the statements to be proven. We then have to include also bounds on momentum derivatives of the Schwinger functions in order to have a complete inductive scheme.

Proposition :
We assume $|w| \le 3$, where the derivatives are taken w.r.t. some momentum p_i. Furthermore $j \in \{1, \ldots, 2n\}/\{i\}$. There exists a constant $K > 0$ such that for $2n > 4$

$$|\partial^w \mathcal{L}^{\Lambda,\Lambda_0}_{2n,l}(\vec{p})| \le \kappa^{4-2n}\, \frac{K^{2l+n-2}}{(l+1)^2\, n!\, n^3}\, (n+l-1)!\, \frac{1}{\big(\sup(\kappa, \eta^{(2n)}_{i,j})\big)^{|w|}} \sum_{\lambda=0}^{\lambda=l} \frac{\log^\lambda\big(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\big)}{2^\lambda\, \lambda!}\,. \tag{46}$$

For $2n = 4$, $|w| \ge 1$

$$|\partial^w \mathcal{L}^{\Lambda,\Lambda_0}_{4,l}(\vec{p})| \le \frac{K^{2l-1/4}}{(l+1)^2\, 2^4}\, (1+l)!\, \frac{1}{\big(\sup(\kappa, \eta^{(4)}_{i,j})\big)^{|w|}} \sum_{\lambda=0}^{\lambda=l-1} \frac{\log^\lambda\big(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\big)}{2^\lambda\, \lambda!}\Big)\,. \tag{47}$$

For $2n = 4$, $|w| = 0$

$$|\mathcal{L}^{\Lambda,\Lambda_0}_{4,l}(\vec{p})| \le \frac{K^{2l}}{(l+1)^2\, 2^4}\, (1+l)! \sum_{\lambda=0}^{\lambda=l-1} \frac{\log^\lambda\big(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\big)}{2^\lambda\, \lambda!} \Big(1 + \log\big(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\big)\Big)\,, \tag{48}$$

$$|\mathcal{L}^{\Lambda,\Lambda_0}_{4,l}(0, p_2, p_3)| \le \frac{K^{2l}}{(l+1)^2\, 2^4}\, (1+l)! \sum_{\lambda=0}^{\lambda=l} \frac{\log^\lambda\big(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\big)}{2^\lambda\, \lambda!}\,. \tag{49}$$

For $2n = 2$, $|w| = 3$

$$|\partial^w \mathcal{L}^{\Lambda,\Lambda_0}_{2,l}(p)| \le \sup(|p|,\kappa)^{-1}\, \frac{K^{2l-1-\frac{1}{4}}}{(l+1)^2}\, l! \sum_{\lambda=0}^{\lambda=l-2} \frac{\log^\lambda\big(\sup(\frac{|p|}{\kappa}, \frac{\kappa}{m})\big)}{2^\lambda\, \lambda!}\,. \tag{50}$$

For $2n=2\,,\ 0\le|w|\le 2\,,\ l\ge 2$

$$|\partial^w \mathcal{L}_{2,l}^{\Lambda,\Lambda_0}(p)| \le \sup(|p|,\kappa)^{2-|w|}\ \frac{K^{2l-1}}{(l+1)^2}\ l!\ \sum_{\lambda=0}^{\lambda=l-2} \frac{\log^\lambda\big(\sup(\frac{|p|}{\kappa},\frac{\kappa}{m})\big)}{2^\lambda\,\lambda!}\ \Big(1+\log(\sup(\frac{|p|}{\kappa},\frac{\kappa}{m}))\Big)\ . \tag{51}$$

For $2n=2\,,\ |w|\in\{0,\,2\}\,,\ l\ge 2$

$$|\partial^w \mathcal{L}_{2,l}^{\Lambda,\Lambda_0}(0)| \le \kappa^{2-|w|}\ \frac{K^{2l-1}}{(l+1)^2}\ l!\ \sum_{\lambda=0}^{\lambda=l-1} \frac{\log^\lambda\big(\frac{\kappa}{m}\big)}{2^\lambda\,\lambda!}\ . \tag{52}$$

Remarks :
Note that j in (46) - (47) is otherwise arbitrary apart from the condition $j\neq i$, so that the bound arrived at will be in fact

$$|\partial^w \mathcal{L}_{2n,l}^{\Lambda,\Lambda_0}(\vec{p})| \le$$

$$\kappa^{4-2n}\ \frac{K^{2l+n-2}}{(l+1)^2\,n!\,n^3}\ \inf_{j,1\le j\le 2n} \frac{1}{\big(\sup(\kappa,\eta_{i,j}^{(2n)})\big)^{|w|}}(n+l-1)!\sum_{\lambda=0}^{\lambda=l} \frac{\log^\lambda\big(\sup(\frac{|\vec{p}|}{\kappa},\frac{\kappa}{m})\big)}{\lambda!}\ .$$

We will choose $j=2n$ in the proof. This means that the momentum p_{2n} will be eliminated on both sides of the FE.
Since the elementary vertex has a weight $\frac{g}{4!}$, a perturbative Schwinger function $\mathcal{L}_{2n,l}$ carries a factor $(\frac{g}{4!})^{l+n-1}$ For simplicity of notation *we replace this factor by one in the subsequent proof.* So the final numerical bound on the Schwinger functions in terms of the constant K, see (76) below, should be multiplied by this factor.

Proof :
The above described inductive scheme starts from the constant $\mathcal{L}_{4,0}^{\Lambda,\Lambda_0}$ at loop order 0. From this term, irrelevant tree level terms with $n>2$ are produced by the second term on the r.h.s. of the FE. For those terms the Proposition is verified from a simplified version of part A) II) of the proof, where all sums over loops are suppressed. Note also that the two-point function for $l=1$ is given by the momentum independent tadpole which is bounded by κ^2. We will subsequently assume that $l\ge 1$ for simplicity of notation.

A) **Irrelevant terms** with $2n+|w|\ge 5$:

I) *The first term on the r.h.s. of the FE*

a) $2n>4$:

Integrating the FE (7) w.r.t. the flow parameter κ' from κ to Λ_0+m gives the following bound for the first term on the r.h.s. of the FE - denoting $\Lambda'=\kappa'-m$ and, as a shorthand,

$|\vec{p}|_{2n+2} = \sup(|\vec{p}|,\, |k|,\, |-k|) = \sup(|p_1|, \ldots, |p_{2n}|,\, |k|)\,,\ \eta^{(2n+2)}_{i,2n} = \eta^{(2n+2)}_{i,2n}(\vec{p}, k, -k)\quad :$

$$\frac{(2n+1)(2n+2)}{2} \int_\kappa^{\Lambda_0+m} d\kappa' \int_k \frac{2}{\Lambda'^3}\ e^{-\frac{k^2+m^2}{\Lambda'^2}}\ \kappa'^{4-(2n+2)}\ \frac{K^{2l+n-3}}{l^2\,(n+1)!\,(n+1)^3}$$

$$\times\,(n+l-1)!\ \frac{1}{\big(\sup(\kappa', \eta^{(2n+2)}_{i,2n})\big)^{|w|}} \sum_{\lambda=0}^{\lambda=l-1} \frac{\log^\lambda(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m}))}{2^\lambda\,\lambda!}$$

$$\leq\ (\frac{n}{n+1})^3\ (2n+1)\ \frac{K^{2l+n-3}}{l^2\,n!\,n^3}\ (n+l-1)! \sum_{\lambda=0}^{\lambda=l-1} \frac{1}{2^\lambda\,\lambda!} \tag{53}$$

$$\times\ K_2 \int_\kappa^{\Lambda_0+m} d\kappa'\ \kappa'^{3-2n-|w|} \int_k \frac{1}{\kappa'^4}\ \frac{1}{\big(\sup(1, \frac{\eta^{(2n+2)}_{i,2n}}{\kappa'})\big)^{|w|}}\ e^{-\frac{k^2}{\Lambda'^2}}\ \log^\lambda\Big(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m})\Big)\ .$$

We used Lemma 6, (31). We bound the momentum integral as follows, setting $x = \frac{k}{\kappa'}$:

$$\int_x \frac{1}{\big(\sup(1, \frac{\eta^{(2n+2)}_{i,2n}}{\kappa'})\big)^{|w|}}\ e^{-x^2}\ \log^\lambda\Big(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m})\Big) \leq$$

$$\sup_x\Big\{e^{-\frac{x^2}{2}}\ \frac{1}{\big(\sup(1, \frac{\eta^{(2n+2)}_{i,2n}}{\kappa'})\big)^{|w|}}\Big\} \int_x e^{-\frac{x^2}{2}}\ \log^\lambda\Big(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m})\Big)\ . \tag{54}$$

The first term is bounded[6] with the aid of Lemma 3, (23), as

$$\sup_x\Big\{e^{-\frac{x^2}{2}}\ \frac{1}{\big(\sup(1, \frac{\eta^{(2n+2)}_{i,2n}}{\kappa'})\big)^{|w|}}\Big\}\ \leq\ c(|w|)\ \frac{1}{\big(\sup(1, \frac{\eta^{(2n)}_{i,2n}}{\kappa'})\big)^{|w|}}\ .$$

To bound the integral in (54), we note that

$$\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'}, \frac{\kappa'}{m})\ \leq\ \sup(\frac{|\vec{p}|}{\kappa'} + \frac{|k|}{\kappa'}, \frac{\kappa'}{m})$$

so that the integral can be bounded using

$$\int_x e^{-\frac{x^2}{2}}\ \log^\lambda(\sup(|x|+a, b))\ \leq\ \int_x e^{-\frac{x^2}{2}}\ \log^\lambda(|x|+a)\ +\ \int_x e^{-\frac{x^2}{2}}\ \log^\lambda b \tag{55}$$

with $a = \frac{|\vec{p}|}{\kappa'}$ and $b = \frac{\kappa'}{m}$. We have

$$\int_x e^{-\frac{x^2}{2}}\ \log^\lambda b\ =\ \frac{1}{4\pi^2}\ \log^\lambda b\ . \tag{56}$$

[6] by the definition of η (43) we have $\eta^{(2n+2)}_{i,2n} \in \{|q|, |q \pm k|\}$, if $\eta^{(2n)}_{i,2n} = |q|$.

Using Lemma 4, (26) and $\frac{1}{4\pi^2}+\frac{1}{4}\le\frac{1}{3}$, we can then bound the integral from (54) by

$$\int_x e^{-\frac{1}{2}x^2}\ \log^\lambda(\sup(\frac{|\vec{p}|_{2n+2}}{\kappa'},\frac{\kappa'}{m}))\ \le\ K_3\ \Big(\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa'},\frac{\kappa'}{m}))\ +\ [\lambda!]^{1/2}\Big)\ , \tag{57}$$

where

$$K_3\ =\ \frac{1}{3}\ . \tag{58}$$

With these results (53) can now be bounded by

$$(\frac{n}{n+1})^3\ (2n+1)\ \frac{K^{2l+n-3}}{l^2\,n!\,n^3}\ (n+l-1)!\ K_2\ K_3\frac{c(|w|)}{\big(\sup(1,\frac{\eta^{(2n)}_{i,2n}}{\kappa})\big)^{|w|}} \tag{59}$$

$$\times\ \sum_{\lambda=0}^{\lambda=l-1}\int_\kappa^{\Lambda_0+m}d\kappa'\ \kappa'^{3-2n-|w|}\ \frac{1}{2^\lambda\,\lambda!}\Big(\log^\lambda(\sup(\frac{|\vec{p}|}{\kappa'},\frac{\kappa'}{m}))\ +\ [\lambda!]^{1/2}\Big)\ .$$

Using Lemma 5, (29) we find - writing $s=2n+|w|-4$ -

$$\sum_{\lambda=0}^{\lambda=l-1}\frac{1}{2^\lambda\,\lambda!}\ \int_\kappa^{\Lambda_0+m}d\kappa'\ \kappa'^{-s-1}\ \Big(\log^\lambda(\sup(\frac{|\vec{p}|_{2n}}{\kappa'},\frac{\kappa'}{m}))\ +\ [\lambda!]^{1/2}\Big)$$

$$\le\ \frac{\kappa^{-s}}{s}\Big\{\ 3\ \sum_{\lambda=0}^{\lambda=l-1}\frac{1}{2^\lambda\,\lambda!}\ \log^\lambda\sup(\frac{|\vec{p}|}{\kappa},\frac{\kappa}{m})\ +\ 2\ \Big\}\le\ 5\ \frac{\kappa^{-s}}{s}\sum_{\lambda=0}^{\lambda=l-1}\frac{1}{2^\lambda\,\lambda!}\ \log^\lambda\sup(\frac{|\vec{p}|}{\kappa},\frac{\kappa}{m})\ .$$

Using this bounds in (59), the first term on the r.h.s. of the FE then satisfies the induction hypothesis (46)[7],

$$\kappa^{4-2n}\ \frac{K^{2l+n-2}}{(l+1)^2\,n!\,n^3}\ (n+l-1)!\ \frac{1}{\big(\sup(\kappa,\eta^{(2n)}_{i,2n})\big)^{|w|}}\sum_{\lambda=0}^{\lambda=l-1}\frac{1}{2^\lambda\,\lambda!}\ \log^\lambda\sup(\frac{|\vec{p}|}{\kappa},\frac{\kappa}{m})\ ,$$

on imposing the lower bound on K

$$K^{-1}\,(\frac{n}{n+1})^3\ (2n+1)\,\frac{(l+1)^2}{l^2}\ K_2\ K_3\ c(|w|)\ \frac{5}{(2n+|w|-4)}\ \le\ 1\ . \tag{60}$$

b) $2n=4\,,\ |w|\ge 1$:

The only change w.r.t. part a) is that we have to verify the bound with an addditional factor of $K^{-1/4}$ appearing in (47). We therefore arrive at the bound

$$K^{-\frac{3}{4}}\ (\frac{2}{3})^3\ 5\ \frac{(l+1)^2}{l^2}\ K_2\ K_3\ c(|w|)\,\frac{5}{|w|}\ \le\ 1\ . \tag{61}$$

[7] we may note that for this term the sum extends up to $l-1$ only

c) $2n = 2\,,\ |w| = 3$:

Due to the momentum derivatives the corresponding contribution for $l = 1$ vanishes. Using the induction hypothesis on $|\partial^w \mathcal{L}^{\Lambda,\Lambda_0}_{4,l-1}(\vec{p})|$ for $l \geq 2$ as in (53) we obtain in close analogy with A) I) a) and b) the following bound

$$(\frac{1}{2})^3\ \frac{K^{2l-1-\frac{1}{4}}}{(l+1)^2}\ \frac{\kappa^2}{\sup(|p|,\,\kappa)^3}\ l!\sum_{\lambda=0}^{\lambda=l-2}\frac{\log^\lambda\bigl(\sup(\frac{|p|}{\kappa},\frac{\kappa}{m})\bigr)}{2^\lambda\,\lambda!}$$

in agreement with (50), on imposing the lower bound

$$K^{-1}\ \frac{3}{8}\ \frac{(l+1)^2}{l^2}\ K_2\ K_3\ c(3)\ 5\ \leq\ 1\ . \tag{62}$$

II) *The second term on the r.h.s. of the FE*

a) $2n > 4$:

We sum over all contributions without taking into account the fact that some of them are suppressed by supplementary fractional powers of K. Some additional precaution is required in the presence of relevant terms, i.e. underived four-point functions, and two-point functions derived at most twice. These functions are decomposed as

$$\mathcal{L}_{4,l}(p_1,p_2,p_3)\ =\ \mathcal{L}_{4,l}(0,p_2,p_3)\ +\ p_{1,\mu}\int_0^1 d\tau\ \partial_{1,\mu}\,\mathcal{L}_{4,l}(\tau p_1,p_2,p_3)\ . \tag{63}$$

For the two-point function we may suppose without limitation that $p = (p_0, 0, 0, 0)$. We then write p instead of p_0, ∂ instead $\frac{\partial}{\partial p}$ and interpolate

$$\partial^2\mathcal{L}_{2,l}(p) = \partial^2\mathcal{L}_{2,l}(0)\ +\ p\int_0^1 d\tau\ \partial^3\mathcal{L}_{2,l}(\tau p)\ , \tag{64}$$

$$\partial\mathcal{L}_{2,l}(p) = p\,\partial^2\mathcal{L}_{2,l}(0)\ +\ p^2\int_0^1 d\tau\ (1-\tau)\ \partial^3\mathcal{L}_{2,l}(\tau p)\ , \tag{65}$$

$$\mathcal{L}_{2,l}(p) = \mathcal{L}_{2,l}(0)\ +\ \frac{1}{2}\,p^2\,\partial^2\mathcal{L}_{2,l}(0)\ +\ p^3\int_0^1 d\tau\ \frac{(1-\tau)^2}{2!}\ \partial^3\mathcal{L}_{2,l}(\tau p)\ . \tag{66}$$

In case of the four-point function we use the bound from (49) for the first term of the decomposition, and the bound from (47) for the second term. Here the interpolated momentum p_1 will be (without loss of generality) supposed to be the momentum q of the propagator linking the two terms on the r.h.s. of the FE. We then will use the bound (35) to get rid of the momentum factor produced through interpolation. *Thus we can avoid*

using (48) which would not reproduce a bound matching with our induction hypothesis. For the two-point function we similarly use either the bounds (52) at zero momentum, or (50), together with (36) and (31), for the interpolated term.

These decompositions lead to additional factors in the bounds. So as not to produce too lengthy expressions we will first write the bounds only for the contributions where the additonal factors are not present and add the modifications necessitated by those terms afterwards (see after (74)).

A second point has to be clarified (which is treated in a fully explicit though notationally more complex way in [GK]). When deriving both sides of the flow equation w.r.t. the momentum p_i, there may arise two situations for the second term on the r.h.s. : either the two momenta p_i and p_{2n} appear both as external momenta of only one term $\mathcal{L}_{n_i,l_i}$, or each of them appears in a different $\mathcal{L}_{n_i,l_i}$. In the first case the derivatives only apply to the term where they both appear, and not to the second one which is independent of p_i, nor to the propagator linking the two terms. In the second case also the other term and the linking propagator depend on p_i via the momentum q of the propagator which is a subsum of momenta containing p_i. Applying then the induction hypothesis to both terms we get a product of η-terms which can be bounded by a single one :

$$\frac{1}{\big(\sup(\kappa,\eta_{i,2n_1}^{(2n_1)})\big)^{|w_1|}}\ \frac{1}{\big(\sup(\kappa,\eta_{i,2n}^{(2n_2)})\big)^{|w_2|}}\ \le\ \frac{1}{\big(\sup(\kappa,\eta_{i,2n}^{(2n)})\big)^{|w_1|+|w_2|}}\ , \tag{67}$$

since one verifies that the set of momenta over which the inf is taken in η in the terms on the l.h.s. of (67) is contained in the one on the r.h.s. of (67). Here $\eta_{i,2n_1}^{(2n_1)}$ has been introduced as in (43) for the momentum set $\{p_1,\ldots,p_{2n_1-1},q\}$, where $q=-p_1-p_2-\ldots-p_{2n_1-1}$, and we understand (without introducing new notation) that $\eta_{i,2n}^{(2n_2)}$ has been introduced as in (43) for the momentum set $\{q,p_{2n_1},\ldots,p_{2n}\}$ where q takes the role of p_i. The reasoning remains the same, if permutations of these momentum sets are considered, which still leave p_i and p_{2n} in different sets.

Integrating the inductive bound on the *second* term on the r.h.s. of the FE from κ to Λ_0+m then gives us the following bound - where we also understand that the sup w.r.t. the previously mentioned permutations has been taken for the momentum attributions

$$\int_\kappa^{\Lambda_0+m} d\kappa'\ \kappa'^{8-(2n+2)}\ K^{2l+n-3} \sum_{\substack{l_1+l_2=l,\\ w_1+w_2+w_3=w,\\ n_1+n_2=n+1}} 2\,c_{\{w_i\}}\ \frac{n_1}{(l_1+1)^2\,n_1!\,n_1^3}\ \frac{n_2}{(l_2+1)^2\,n_2!\,n_2^3}$$

$$\times\ \frac{1}{\big(\sup(\kappa',\eta_{i,2n_1}^{(2n_1)})\big)^{|w_1|}}\ (n_1+l_1-1)! \sum_{\lambda_1=0}^{\lambda_1=l_1} \frac{\log^{\lambda_1}\big(\sup(\frac{|\vec{p}|}{\kappa'},\frac{\kappa'}{m})\big)}{2^{\lambda_1}\,\lambda_1!}\ \frac{2}{\Lambda'^3}\ |\partial^{w_3}\ e^{-\frac{q^2+m^2}{\Lambda'^2}}|$$

$$\times \frac{1}{\big(\sup(\kappa',\eta^{(2n)}_{i,j_2})\big)^{|w_2|}}\ (n_2+l_2-1)!\ \sum_{\lambda_2=0}^{\lambda_2=l_2} \frac{\log^{\lambda_2}\big(\sup(\frac{|\vec p|}{\kappa'},\frac{\kappa'}{m})\big)}{2^{\lambda_2}\,\lambda_2!}\ .$$

We use (67) to bound the previous expression by

$$\sum_{\substack{l_1+l_2=l,\\ n_1+n_2=n+1,\\ \lambda_1\le l_1,\,\lambda_2\le l_2}} \frac{1}{(l_1+1)^2\,(l_2+1)^2}\ \frac{1}{n_1^2\,n_2^2}\ \frac{n!}{n_1!\,n_2!}\ \frac{(\lambda_1+\lambda_2)!}{\lambda_1!\,\lambda_2!}\ \frac{(n_1+l_1-1)!\ (n_2+l_2-1)!}{(n+l-1)!}$$

$$\times\ 2\ K^{2l+n-3}\ \frac{(n+l-1)!}{n!}\int_\kappa^{\Lambda_0+m} d\kappa'\ \kappa'^{3-2n}\ \frac{\log^{\lambda_1+\lambda_2}\big(\sup(\frac{|\vec p|}{\kappa'},\frac{\kappa'}{m})\big)}{2^{\lambda_1+\lambda_2}\,(\lambda_1+\lambda_2)!}$$
$$\times \sum_{w_1+w_2+w_3=w} c_{\{w_i\}}\ \frac{2}{\Lambda'^3}\ |\partial^{w_3}\ e^{-\frac{q^2+m^2}{\Lambda'^2}}|\ \frac{1}{\big(\sup(\kappa',\eta^{(2n)}_{i,2n})\big)^{|w_1|+|w_2|}}\ .$$

Using Lemma 2, (15) and Lemma 6, (33), and the fact that

$$\sup(|q|,\kappa')^{-|w_3|}\ \frac{1}{\big(\sup(\kappa',\eta^{(2n)}_{i,2n})\big)^{|w_1|+|w_2|}}\ \le\ \frac{1}{\big(\sup(\kappa',\eta^{(2n)}_{i,2n})\big)^{|w|}}$$

we then arrive at the bound

$$K_0\ \frac{1}{(l+1)^2}\ \frac{1}{n^2}\ 2\ K^{2l+n-3}\ \frac{1}{n!}\ (n+l-1)!\int_\kappa^{\Lambda_0+m} d\kappa'\ \kappa'^{3-2n-|w|}\ \sum_{0\le\lambda\le l}\frac{\log^{\lambda}\big(\sup(\frac{|\vec p|}{\kappa'},\frac{\kappa'}{m})\big)}{2^{\lambda}\,\lambda!}$$
$$\times\sum_{w_i} c_{\{w_i\}}\,K^{(|w_3|)}\ \frac{1}{\big(\sup(1,\frac{\eta^{(2n)}_{i,2n}}{\kappa'})\big)^{|w|}}\ . \tag{68}$$

Using also Lemma 5 we verify the bound (46)

$$\kappa^{4-2n}\ K^{2l+n-2}\ \frac{1}{(l+1)^2}\ \frac{1}{n^3}\frac{1}{n!}\ (n+l-1)!\sum_{0\le\lambda\le l}\frac{\log^{\lambda}\big(\sup(\frac{|\vec p|}{\kappa},\frac{\kappa}{m})\big)}{2^{\lambda}\,\lambda!}\ \frac{1}{\big(\sup(\kappa,\eta^{(2n)}_{i,j})\big)^{|w|}}\ ,$$

on imposing the lower bound on K

$$K^{-1}\ 3\cdot 2\ K_2\ \frac{n}{2n+|w|-4}\ K_0\sum_{w_i} c_{\{w_i\}}\,K^{(|w_3|)}\ \le\ 1\ ,\quad n>2\ . \tag{69}$$

b) $2n=4, |w|\ge 1$:

We obtain in the same way, using Lemma 2c)

$$K^{-3/4}\ 6\ K_2\ 2\ K_0''\sum_{\{w_i\}} c_{\{w_i\}}\,K^{(|w_3|)}\ \le 1\ . \tag{70}$$

c) $2n = 2, |w| = 3$:

For the two-point function we obtain

$$K^{-3/4}\; 6\; K_2\; K_0''\sum_{\{w_i\}} c_{\{w_i\}}\, K^{(|w_3|)} \;\le 1\ . \tag{71}$$

Taking both contributions from the r.h.s. of the FE together, the lower bounds on K become for $n > 2$

$$K_2\,\Big(5\; K_3\;(\frac{n}{n+1})^3\;\frac{c(|w|)\,(2n+1)\,(l+1)^2}{(2n+|w|-4)\,l^2}\;+\;\frac{6\;n}{2n+|w|-4}\;K_0\sum_{\{w_i\}} c_{\{w_i\}}\,K^{(|w_3|)}\Big)\;\le K\ , \tag{72}$$

and for $n = 2$ resp. $n = 1$

$$K_2\,\Big(5\cdot 5\;(\frac{2}{3})^3\;K_3\;c(|w|)\,\frac{l+1)^2}{|w|\,l^2}\;+\;6\cdot 2\;K_0''\;\frac{2}{|w|}\sum_{\{w_i\}} c_{\{w_i\}}\,K^{(|w_3|)}\Big)\;\le K^{\frac{3}{4}}\ , \tag{73}$$

$$K_2\,\Big(5\cdot\frac{3}{8}\;K_3\;c(3)\;\frac{(l+1)^2}{l^2}\;K^{-\frac{1}{4}}\;+\;6\;K_0''\sum_{\{w_i\}} c_{\{w_i\}}\,K^{(|w_3|)}\Big)\;\le K^{\frac{3}{4}}\ . \tag{74}$$

We now come back to the modifications required because of the decompositions (63), (64), (65), (66). We introudce the shorthands $\sum_{\{w_i\}} c_{\{w_i\}}\,K^{(|w_3|)} \equiv \tilde K(w) \equiv \tilde K$ and $\sum_{\{w_i\}} c_{\{w_i\}}\,K'^{(|w_3|)} \equiv \tilde K'(w) \equiv \tilde K'$. In order not to inflate too much the values of the constants we distinguish different cases. In each case we have to replace the factors $K_0\,\tilde K$ from (69) resp. $K_0''\,\tilde K$ from (70) and from (71) by the following ones:

i) $n > 3$:

$$\frac{K_0}{2}\,\tilde K\;+2K_0'\,\tilde K\;+\;2K_0'\,\frac{2}{\sqrt{e}\,K^{1/4}}\,\tilde K'\;+\;2K_0''\,\tilde K\;+\;K_0''\,(\frac{1}{\sqrt{e}}\;+\;\frac{1}{2}\;\frac{2}{e}\;+\;\frac{1}{K^{1/4}})\,\tilde K'\ \ ,$$

ii) $n = 3$:

$$\frac{K_0}{2}\,\tilde K\;+\;K_0'\,\tilde K\;+\;K_0'\,(\frac{2}{\sqrt{e}\,K^{1/4}}\;+\;\frac{2}{e\,K^{1/2}})\,\tilde K'\;+\;2K_0''\,\tilde K\;+\;2K_0''\,(\frac{1}{\sqrt{e}}\;+\;\frac{1}{2}\;\frac{2}{e}\;+\;\frac{1}{K^{1/4}})\,\tilde K'\,,$$

iii) $n = 2$:

$$2\;K_0''\,\tilde K\;+\;2\;K_0''\,\frac{1}{\sqrt{e}\,K^{1/4}}2\;\;\tilde K'\ ,$$

iv) $n = 1$:

$$K_0''\,\tilde K\;+\;K_0''\,(\frac{1}{\sqrt{e}}\;+\;\frac{1}{2}\;\frac{2}{e}\;+\;\frac{1}{K^{1/4}})\,\tilde K'\ .$$

These factors can be understood as follows :
In case i) we may replace K_0 by $K_0/2$ if no two- or four-point functions appear by Lemma 2, (16). In the other cases we use Lemma 2, (17) or(18), and we use the decompositions which then give rise to a sum of contributions. Factors of 2 appear if there exist two contributions of the required type. To bound the individual terms from the decomposition we also have to use Lemma 6 c), since there appear momentum dependent factors in the interpolation formulas which have to be bounded with the aid of the regularizing exponential. The terms multiplied by $\tilde{K}$ thus arise from the boundary terms, those multiplied by $\tilde{K}'$ from interpolated ones where the bounds (34) instead of (33) have to be used since the regularizing exponential has to be split up and used for bounding two types of momentum factors. In the cases $n = 2$ and $n = 1$ there appear one four- and on two-point function resp. two two-point functions on the r.h.s. of the FE. Only one of these factors has to be decomposed however, since in the final bound we can tolerate *one* factor of $(1+\log\big(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\big)$ according to the induction hypotheses for these two cases, see (48), (51).

The final lower bound on K which also turns out to be the most stringent one in the end, stems from the case $n = 3$. It is thus the following one

$$\Big\{5K_3\,(\frac{3}{4})^3\,\frac{c(|w|)\,7\,(l+1)^2}{l^2} \,+\, 18\,\Big[\frac{K_0}{2}\,\tilde{K} \,+\, K_0'\,\tilde{K} \,+\, K_0'\,(\frac{2}{\sqrt{e}\,K^{1/4}} \,+\, \frac{2}{e\,K^{1/2}})\,\tilde{K}'$$

$$+\, 2K_0''\,\tilde{K} \,+\, 2K_0''\,(\frac{1}{\sqrt{e}} \,+\, \frac{1}{2}\,\frac{2}{e} \,+\, \frac{1}{K^{1/4}})\,\tilde{K}'\Big]\Big\}\,\frac{K_2}{2+|w|} \;\le\; K\,. \tag{75}$$

The numerical lower bound on K deduced from (75) in the worst case $|w| = 3$ is

$$K \;\ge\; 6.2\cdot 10^5\,. \tag{76}$$

One could certainly gain several orders of magnitude by more carefully bounding individual special cases (see above for one point). *The basic source of the (still) large numerical constant is in the fact that we have to reconstruct the relevant terms from their derivatives.*

B) **Relevant terms** with $2n + |w| \le 4$:
a) $2n = 4\,,\ |w| = 0$:
We first look at $\mathcal{L}^{\Lambda,\Lambda_0}_{4,l}(\vec{0})$ which is decomposed as

$$\mathcal{L}^{\Lambda,\Lambda_0}_{4,l}(\vec{0}) \;=\; \mathcal{L}^{0,\Lambda_0}_{4,l}(\vec{0}) \,+\, \int_0^\Lambda d\Lambda'\;\partial_{\Lambda'}\mathcal{L}^{\Lambda',\Lambda_0}_{4,l}(\vec{0})\;, \tag{77}$$

where the first term is vanishes for $l \geq 1$, see (42). For the second term we obtain by induction from the *first* term on the r.h.s. of the FE the bound

$$\binom{6}{2} \int_m^{\Lambda+m} d\kappa' \int_k \frac{2}{\Lambda'^3}\, e^{-\frac{k^2+m^2}{\Lambda'^2}}\, \kappa'^{-2}\, \frac{K^{2l-1}}{l^2\, 2\cdot 3^4}\, (1+l)! \sum_{\lambda=0}^{\lambda=l-1} \frac{\log^\lambda(\sup(\frac{|k|}{\kappa'},\frac{\kappa'}{m}))}{2^\lambda\, \lambda!}$$

$$\leq\; K_2\, K_3\, \binom{6}{2} \frac{1}{2\cdot 3^4}\, \frac{K^{2l-1}}{l^2}\, (1+l)! \sum_{\lambda=0}^{\lambda=l-1} \frac{1}{2^\lambda\, \lambda!} \int_m^{\Lambda+m} d\kappa'\, \kappa'^{-1} \Big(\log^\lambda(\frac{\kappa'}{m}) + (\lambda!)^{1/2} \Big)\,, \tag{78}$$

where we used again (31) and (57), remembering that $|\vec{p}| = |k|$ in the present case. We have

$$\int_m^\kappa \frac{d\kappa'}{\kappa'} \Big(\log^\lambda(\frac{\kappa'}{m}) \,+\, [\lambda!]^{1/2} \Big) \;=\; \frac{\log^{\lambda+1}(\frac{\kappa}{m})}{\lambda+1} \,+\, \log(\frac{\kappa}{m})\, [\lambda!]^{1/2}\,, \tag{79}$$

$$\sum_{\lambda=0}^{\lambda=l-1} \Big(\frac{\log^{\lambda+1}(\frac{\kappa}{m})}{2^\lambda\, (\lambda+1)!} + \log(\frac{\kappa}{m})\, \frac{1}{2^\lambda\, \lambda!^{1/2}} \Big) \;\leq\; \inf\Big\{ 6 \sum_{\lambda=1}^{\lambda=l} \frac{\log^\lambda(\frac{\kappa}{m})}{2^\lambda\, \lambda!}\,,\; 2 \sum_{\lambda=0}^{\lambda=l-1} \frac{\log^\lambda(\frac{\kappa}{m})}{2^\lambda\, \lambda!} (1+\log\frac{\kappa}{m}) \Big\}\,. \tag{80}$$

Using the first of these bounds in (78), the first term on the r.h.s. of the FE is bounded in agreement with the induction hypothesis by

$$\frac{K^{2l}}{(l+1)^2\, 2^4}\, (1+l)! \sum_{\lambda=0}^{\lambda=l} \frac{\log^\lambda(\frac{\kappa}{m})}{2^\lambda\, \lambda!}\,, \tag{81}$$

assuming the lower bound on K

$$K^{-1}\; 6\; K_2\; K_3\; \binom{6}{2}\, \frac{2^4}{2\cdot 3^4}\, \frac{(l+1)^2}{l^2} \;\leq\; 1\,. \tag{82}$$

In the contribution from the *second* term on the r.h.s. of the FE we have one contribution with $n_1 = 2$ and one contribution with $n_2 = 1$ or vice versa. Integrating the FE (7) w.r.t. the flow parameter at vanishing momentum gives the inductive bound, using (49), (52) and Lemma 2 c)

$$2\cdot 4 \int_m^{\Lambda+m} d\kappa'\, \frac{2}{\Lambda'^3}\, e^{-\frac{m^2}{\Lambda'^2}}\, \kappa'^2\, K^{2l-1} \sum_{\substack{l_1+l_2=l,\\ l_2\geq 1}} \frac{(1+l_1)!}{(l_1+1)^2\, 2^4}\, \frac{l_2!}{(l_2+1)^2} \sum_{\lambda_1=0}^{\lambda_1=l_1} \frac{\log^{\lambda_1}(\frac{\kappa'}{m})}{2^{\lambda_1}\, \lambda_1!} \sum_{\lambda_2=0}^{\lambda=l_2-1} \frac{\log^{\lambda_2}(\frac{\kappa'}{m})}{2^{\lambda_2}\, \lambda_2!}$$

$$\leq\; \frac{16\, K^{2l-1}\, K_0''}{(l+1)^2\, 2^4}\, (1+l)! \int_0^\Lambda \frac{d\Lambda'}{\Lambda'^3}\, e^{-\frac{m^2}{\Lambda'^2}}\, \kappa'^2 \sum_{\lambda=0}^{\lambda=l-1} \frac{\log^\lambda(\frac{\kappa'}{m})}{2^\lambda\, \lambda!}\,.$$

With the aid of Lemma 7 a) the previous expression can be bounded as in (81) assuming

$$16\ K_0''\ K_1\ \le\ K\ .$$

To go away from the renormalization point we proceed as in [KM]. In fact, we will distinguish four different situations as regards the momentum configurations. The bounds established in part A) for the case $n=4, |w|=1$ are in terms of the functions $\eta_{i,j}^{(4)}$ from (43). Assuming (without loss of generality)

$$|p_4| \ge |p_1|\ ,\ |p_2|\ ,\ |p_3|\ ,$$

we realize that $\eta_{i,4}^{(4)}$ is always given by a sum of *at most two* momenta from the set $\{p_1\ ,\ p_2\ ,\ p_3\}$. It is then obvious that the subsequent cases ii) and iv) cover all possible situations. The cases i) and iii) correspond to exceptional configurations for which the bound has to be established before proceeding to the general ones. The four cases are

i) $\{p_1\ ,\ p_2\ ,\ p_3\}=\{0\ ,\ q\ ,\ v\}$

ii) $\{p_1\ ,\ p_2\ ,\ p_3\}$ such that $\inf_i \eta_{i,4}^{(4)}\ =\ \inf_i |p_i|$

iii) $\{p_1\ ,\ p_2\ ,\ p_3\}=\{p\ ,-p\ ,\ v\}$

iv) $\{p_1\ ,\ p_2\ ,\ p_3\}$ such that $\inf_i \eta_{i,4}^{(4)}\ =\ \inf_{j\neq k} |p_j+p_k|\ .$

i) To prove the proposition in this case, i. e. (49), we bound

$$|\mathcal{L}_{4,l}^{\Lambda,\Lambda_0}(0,q,v)| \le$$

$$|\mathcal{L}_{4,l}^{\Lambda,\Lambda_0}(0,0,0)| + \sum_\mu \int_0^1 d\tau\ \Big(|q_\mu \partial_{q_\mu} \mathcal{L}_{4,l}^{\Lambda,\Lambda_0}(0,\tau q,\tau v)| +\ |v_\mu \partial_{v_\mu} \mathcal{L}_{4,l}^{\Lambda,\Lambda_0}(0,\tau q,\tau v)|\Big)\ .$$

The second term is bounded using the induction hypothesis:

$$\frac{K^{2l-\frac14}}{(l+1)^2\,2^4} \sum_{i=2,3} |p_i| \int_0^1 d\tau\, \frac{1}{\sup(\kappa,\eta_{i,4}^{(4)}(\tau)\,)}\ (1+l)! \sum_{\lambda=0}^{\lambda=l-1} \frac{1}{2^\lambda\,\lambda!}\ \log^\lambda\!\big(\sup(\frac{|\vec{p}^{\,\tau}|}{\kappa},\frac{\kappa}{m})\big)\ . \quad (83)$$

We have written $\eta(\tau)$ for the η-parameter in terms of the scaled variables $p_2^\tau=\tau q\,,\ p_3^\tau=\tau v$ and $\vec{p}^{\,\tau}$ for the momentum set $(0,\,p_2^\tau,\,p_3^\tau)$. Using Lemma 8 we find $\eta_{2,4}^{(4)}(\tau)=\tau|q|\,,$ $\eta_{3,4}^{(4)}(\tau)=\tau|v|\,,$ and we thus obtain the following bound for (83) - apart from the prefactor

$$|q|\,\Big(\int_0^{\inf(1,\frac{\kappa}{|q|})} \frac{d\tau}{\kappa} + \int_{\inf(1,\frac{\kappa}{|q|})}^1 \frac{d\tau}{\tau|q|}\Big) \log^\lambda\!\big(\sup(\frac{|\vec{p}^{\,\tau}|}{\kappa},\frac{\kappa}{m})\big) + \Big(q\to v\Big)\ . \qquad (84)$$

If $|q|\ge\kappa$ we find

$$\int_{\frac{\kappa}{|q|}}^1 \frac{d\tau}{\tau}\ \log^\lambda\!\big(\sup(\frac{|\vec{p}^{\,\tau}|}{\kappa},\frac{\kappa}{m})\big) \le \int_{\frac{\kappa}{|\vec{p}|}}^1 \frac{d\tau}{\tau}\ \log^\lambda\!\big(\frac{\tau|\vec{p}|}{\kappa}\big) \le \int_1^{\frac{|\vec{p}|}{\kappa}} \frac{dx}{x}\ \log^\lambda x\ = \frac{\log^{\lambda+1}(\frac{|\vec{p}|}{\kappa})}{\lambda+1}$$

with an analogous calculation for $|v| \geq \kappa$. We thus obtain a bound for (84)

$$2\, \log^\lambda\big(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\big) \,+\, 2\, \frac{\log^{\lambda+1}\big(\sup(1, \frac{|\vec{p}|}{\kappa})\big)}{\lambda+1} \tag{85}$$

which allows to bound (83) by

$$\frac{6\, K^{2l-\frac{1}{4}}}{(l+1)^2\, 2^4}\, (1+l)! \sum_{\lambda=0}^{\lambda=l} \frac{1}{2^\lambda\, \lambda!}\, \log^\lambda\big(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\big)\,. \tag{86}$$

Using this bound together with the previous one on $\mathcal{L}_{4,l}^{\Lambda,\Lambda_0}(0,0,0)$ we verify the induction hypothesis on $\mathcal{L}_{4,l}^{\Lambda,\Lambda_0}(0,q,v)$ (49) under the condition

$$K^{-1}\Big(6\; K_2\; K_3\; \binom{6}{2}\, \frac{2^4}{2\cdot 3^4}\, \frac{(l+1)^2}{l^2} \;+\; 16\; K_0''\; K_1\Big) \;+\; 6\; K^{-1/4} \leq\; 1\,. \tag{87}$$

ii) We assume without loss of generality $\inf_i \eta_{i,4}^{(4)} \;=\; |p_1|$. We use again an integrated Taylor formula along the integration path $(p_1^\tau,\, p_2^\tau,\, p_3^\tau) \,=\, (\tau\, p_1,\, p_2,\, p_3 + (1-\tau)\, p_1)$. By Lemma 8 we find $\eta_{1,4}^{(4)}(\tau) = |p_1^\tau| \,=\, \tau|p_1|,\, \eta_{3,4}^{(4)}(\tau) \geq \tau|p_1|$. The boundary term for $\tau \,=\, 0$ is bounded in i). For the second term we bound

$$|\sum_\mu \int_0^1 d\tau\, \Big(p_{1,\mu}\, (\partial_{p_{1,\mu}} - \,\partial_{p_{3,\mu}})\mathcal{L}(p_1^\tau,\, p_2^\tau,\, p_3^\tau\,)\Big)|$$

$$\begin{aligned}
\leq\; & \frac{K^{2l-\frac{1}{4}}(1+l)!}{(l+1)^2\, 2^4} \sum_{\lambda=0}^{\lambda=l-1} \frac{|p_1|}{2^\lambda\, \lambda!} \int_0^1 d\tau\, \big(\frac{1}{\sup(\kappa, \eta_{1,4}^{(4)}(\tau)\,)} + \frac{1}{\sup(\kappa, \eta_{3,4}^{(4)}(\tau)\,)}\big) \log^\lambda\big(\sup(\frac{|\vec{p}^{\,\tau}|}{\kappa}, \frac{\kappa}{m})\big) \\
\leq\; & \frac{K^{2l-\frac{1}{4}}(1+l)!}{(l+1)^2\, 2^4} \sum_{\lambda=0}^{\lambda=l-1} \frac{2\,|p_1|}{2^\lambda\, \lambda!} \Big(\int_0^{\inf(1,\frac{\kappa}{|p_1|})} \frac{d\tau}{\kappa} + \int_{\inf(1,\frac{\kappa}{|p_1|})}^1 \frac{d\tau}{\tau|p_1|}\Big) \log^\lambda\big(\sup(\frac{|\vec{p}^{\,\tau}|}{\kappa}, \frac{\kappa}{m})\big) \\
\leq\; & 2\, \frac{K^{2l-\frac{1}{4}}(1+l)!}{(l+1)^2\, 2^4} \sum_{\lambda=0}^{\lambda=l-1} \frac{1}{2^\lambda\, \lambda!}\, \big[1 \,+\, \log\big(\sup(1, \frac{|p_1|}{\kappa})\big)\big]\, \log^\lambda\big(\sup(\frac{|\vec{p}^{\,\tau}|}{\kappa}, \frac{\kappa}{m})\big) \\
\leq\; & \frac{2\, K^{2l-\frac{1}{4}}}{(l+1)^2\, 2^4}\, (1+l)! \sum_{\lambda=0}^{\lambda=l-1} \frac{1}{2^\lambda\, \lambda!}\, \log^\lambda\big(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})\big)\, \Big(1 + \log\big(\sup(\frac{\kappa}{m}, \frac{|\vec{p}|}{\kappa})\big)\Big)\,.
\end{aligned} \tag{88}$$

Adding the terms from i) to this term gives the lower bound on K

$$K^{-1}\Big(2\; K_2\; K_3\; \binom{6}{2}\, \frac{2^4}{2\cdot 3^4}\, \frac{(l+1)^2}{l^2} \;+\; 16\; K_0''\; K_1\Big) \;+\; 4\; K^{-1/4} \leq\; 1\,. \tag{89}$$

Here we used the fact that we may bound the term from i) also by (88) instead of (86) if we only want to verify the weaker form of the induction hypothesis valid for general

momenta. At the same time we have replaced a factor of 6 appearing in (82) by a factor of 2, since in the general case we may use the second bound in (80).

iii) We choose the integration path $(p_1^\tau,\, p_2^\tau,\, p_3^\tau\,) = \,(\tau\, p,\, -p,\, v\,)$. Here we assume without restriction that $|v| \le |v-(1-\tau)p|$, otherwise we could interchange the role of v and $p_4 = -v$. The boundary term leads again back to i). The integral $\int_0^1 d\tau$ is cut into four pieces - where the configuration $\kappa < 2|p_1|$ gives the largest contribution :

$$\int_0^1 = \int_0^{\inf(1/2,\frac{\kappa}{|p_1|})} + \int_{\inf(1/2,\frac{\kappa}{|p_1|})}^{1/2} + \int_{1/2}^{\sup(1/2,1-\frac{\kappa}{|p_1|})} + \int_{\sup(1/2,1-\frac{\kappa}{|p_1|})}^{1} .$$

They are bounded in analogy with ii) using $\eta_{1,4}^{(4)}(\tau\,) = \tau|p_1|$ for $\tau \le 1/2$, $\eta_{1,4}^{(4)}(\tau\,) = (1-\tau)|p_1|$ for $\tau \ge 1/2$, relations established with the aid of Lemma 8. We get the bound

$$\frac{K^{2l-\frac14}}{(l+1)^2\,2^4}\,(1+l)!\,\sum_{\lambda=0}^{\lambda=|l-1|}\frac{1}{2^\lambda\,\lambda!}\,\log^\lambda\bigl(\sup(\frac{|\vec p|}{\kappa},\frac{\kappa}{m})\bigr)\,\Bigl(1+2\log\bigl(\sup(1,\frac{|\vec p|}{2\kappa})\bigr)\Bigr) \tag{90}$$

so that verification of (48) requires again the lower bound (89) on K .

iv) We assume without loss $\inf_i \eta_{i,4}^{(4)} = |p_1+p_2|$ and integrate along $(p_1^\tau,\, p_2^\tau,\, p_3^\tau\,) = (p_1,\, -p_1+\tau(p_1+p_2),\, p_3\,)$. The boundary term has been bounded in iii). Using Lemma 8 we find $\inf \eta_{2,4}^{(4)}(\tau) = \tau|p_1+p_2|$, and the integration term is then bounded through

$$|\sum_\mu \int_0^1 d\tau\,\Bigl((p_{1,\mu}+p_{2,\mu})\,\partial_{p_{2,\mu}}\mathcal{L}^{\Lambda,\Lambda_0}(p_1^\tau,\,p_2^\tau,\,p_3^\tau\,)\Bigr)| \le$$

$$\frac{K^{2l-\frac14}(1+l)!}{(l+1)^2\,2^4}\sum_{\lambda=0}^{\lambda=l-1}\frac{|p_1+p_2|}{2^\lambda\,\lambda!}\Bigl(\int_0^{\inf(1,\frac{\kappa}{|p_1+p_2|})}\frac{d\tau}{\kappa}+\int_{\inf(1,\frac{\kappa}{|p_1+p_2|})}^1\frac{d\tau}{\tau|p_1+p_2|}\Bigr)\log^\lambda(\sup(\frac{|\vec{p}^\tau|}{\kappa},\,\frac{\kappa}{m}))$$

which gives as before a bound

$$\frac{K^{2l-\frac14}}{(l+1)^2\,2^4}\,(1+l)!\,\sum_{\lambda=0}^{\lambda=l-1}\frac{1}{2^\lambda\,\lambda!}\,\log^\lambda\bigl(\sup(\frac{|\vec p|}{\kappa},\frac{\kappa}{m})\bigr)\,\Bigl(1+\log\bigl(\sup(1,\frac{|\vec p|}{\kappa})\bigr)\Bigr) \tag{91}$$

so that taking into account the boundary term from iii) we finally require

$$K^{-1}\Bigl(2\;K_2\;K_3\;\binom{6}{2}\;\frac{2^4}{2\cdot 3^4}\;\frac{(l+1)^2}{l^2}\;+\;16\;K_0''\;K_1\Bigr)\;+\;5\;K^{-1/4}\le\;1 \tag{92}$$

to be in agreement with induction.

b) $2n = 2$:

We again use the simplified notation (64) to (66). We will assume that $l \geq 2$. We proceed in descending order of $|w|$ starting from

b1) $|w| = 2$:

$$\partial^2 \mathcal{L}_{2,l}(p) = \partial^2 \mathcal{L}_{2,l}(0) + p \int_0^1 d\tau \; \partial^3 \mathcal{L}_{2,l}(\tau p) \; . \tag{93}$$

We first look at $\partial^2 \mathcal{L}_{2,l}^{\Lambda,\Lambda_0}(0)$ which is decomposed as

$$\partial^2 \mathcal{L}_{2,l}^{\Lambda,\Lambda_0}(0) = \partial^2 \mathcal{L}_{2,l}^{0,\Lambda_0}(0) + \int_0^\Lambda d\Lambda' \; \partial_{\Lambda'} \partial^2 \mathcal{L}_{2,l}^{\Lambda',\Lambda_0}(0) \; ,$$

the second term being obtained from the r.h.s. of the FE, and the first vanishing by (42). The *first* term on the r.h.s. of the FE then gives the bound

$$\binom{4}{2} \int_m^{\Lambda+m} d\kappa' \int_k \frac{2}{\Lambda'^3} \, e^{-\frac{k^2+m^2}{\Lambda'^2}} \, \kappa'^{-2} \, \frac{K^{2l-2-\frac{1}{4}}}{l^2 \, 2^4} \; l! \sum_{\lambda=0}^{\lambda=l-2} \frac{\log^\lambda(\sup(\frac{|k|}{\kappa'}, \frac{\kappa'}{m}))}{2^\lambda \, \lambda!}$$

$$\leq K_2 \, K_3 \, \frac{6}{2^4} \, \frac{K^{2l-2-\frac{1}{4}}}{l^2} \; l! \sum_{\lambda=0}^{\lambda=l-2} \frac{1}{2^\lambda \, \lambda!} \int_m^{\Lambda+m} d\kappa' \; \kappa'^{-1} \left(\log^\lambda(\frac{\kappa'}{m}) + (\lambda!)^{1/2} \right) , \tag{94}$$

where we used again (57) and (31), remembering that $|\vec{p}| = k$ in the present case. Using (79) and (80) (with $l \to l-1$) the first term on the r.h.s. of the FE is then bounded in agreement with the induction hypothesis by

$$\frac{K^{2l-1}}{(l+1)^2 \, 2^4} \; l! \sum_{\lambda=0}^{\lambda=l-1} \frac{\log^\lambda(\frac{\kappa}{m})}{2^\lambda \, \lambda!}$$

under the assumption

$$K^{-5/4} \; K_2 \; K_3 \; 6 \cdot 6 \; \frac{(l+1)^2}{l^2} \; \leq \; 1 \; . \tag{95}$$

This contribution has to be added to the one from the *second* term on the r.h.s. of the FE. We have only contributions with $n_1 = 1$ and $n_2 = 1$. The two momentum derivatives have to apply both to the propagator or both to a function $\mathcal{L}_{2,l}$; all other contributions vanish at zero momentum. For the contribution of the first kind integration of the FE (7) gives the bound

$$8 \int_0^\Lambda \frac{d\Lambda'}{\Lambda'^5} \, e^{-\frac{m^2}{\Lambda'^2}} \, \kappa'^4 \, K^{2l-2} \sum_{\substack{l_1+l_2=l, \\ l_1,l_2 \geq 1}} \frac{l_1!}{(l_1+1)^2} \, \frac{l_2!}{(l_2+1)^2} \sum_{\lambda_1=0}^{\lambda_1=l_1-1} \frac{\log^{\lambda_1}(\frac{\kappa'}{m})}{2^{\lambda_1} \, \lambda_1!} \sum_{\lambda_2=0}^{\lambda=l_2-1} \frac{\log^{\lambda_2}(\frac{\kappa'}{m}))}{2^{\lambda_2} \, \lambda_2!}$$

$$\leq\ 8\,\frac{K^{2l-2}\,K_0''}{(l+1)^2}\ \int_0^\Lambda \frac{d\Lambda'}{\Lambda'^5}\ e^{-\frac{m^2}{\Lambda'^2}}\ \kappa'^4 \sum_{\lambda=0}^{\lambda=l-2} \frac{\log^\lambda(\frac{\kappa'}{m})}{2^\lambda\,\lambda!}\ , \tag{96}$$

where we used (9)and Lemma 2 c). Using also Lemma 7 we obtain the bound

$$16\ \frac{K^{2l-2}\,K_0''}{(l+1)^2}\ K_1' \sum_{\lambda=1}^{\lambda=l-1} \frac{\log^\lambda(\frac{\kappa}{m})}{2^\lambda\,\lambda!}\ . \tag{97}$$

For the contribution of the second kind integration of the FE gives in the same way the bound (again using Lemma 2 c) and Lemma 7)

$$4\int_0^\Lambda \frac{d\Lambda'}{\Lambda'^3}\ e^{-\frac{m^2}{\Lambda'^2}}\ \kappa'^2\ K^{2l-2} \sum_{\substack{l_1+l_2=l,\\ l_1,l_2\geq 1}} \ \frac{l_1!}{(l_1+1)^2}\ \frac{l_2!}{(l_2+1)^2} \sum_{\lambda_1=0}^{\lambda_1=l_1-1} \frac{\log^{\lambda_1}(\frac{\kappa'}{m})}{2^{\lambda_1}\,\lambda_1!} \sum_{\lambda_2=0}^{\lambda=l_2-1} \frac{\log^{\lambda_2}(\frac{\kappa'}{m})}{2^{\lambda_2}\,\lambda_2!}$$

$$\leq\ 4\ \frac{K^{2l-2}\,K_0''}{(l+1)^2}\ \int_0^\Lambda \frac{d\Lambda'}{\Lambda'^3}\ e^{-\frac{m^2}{\Lambda'^2}}\ \kappa'^2 \sum_{\lambda=0}^{\lambda=l-2} \frac{\log^\lambda(\frac{\kappa'}{m})}{2^\lambda\,\lambda!}\ \leq\ 8\ \frac{K^{2l-2}\,K_0''}{(l+1)^2}\ K_1 \sum_{\lambda=1}^{\lambda=l-1} \frac{\log^\lambda(\frac{\kappa}{m})}{2^\lambda\,\lambda!}\ . \tag{98}$$

The sum of this bound and the bounds (95), (97) is compatible with the induction hypothesis (52) under the condition

$$K^{-5/4}\ K_2\ K_3\ 36\ \frac{(l+1)^2}{l^2}\ +\ 8\ K^{-1}(2\,K_0''\ K_1'\ +\ K_0''\ K_1)\ \leq\ 1\ . \tag{99}$$

The second term in (93) is bounded with the aid of the induction hypothesis

$$|p|\int_0^1 d\tau\,\partial^3\mathcal{L}_{2,l}(\tau p)|\ \leq\ |p|\int_0^1 \frac{d\tau}{\sup(\tau|p|,\kappa)}\ \frac{K^{2l-1-\frac14}}{(l+1)^2}\ l!\ \sum_{\lambda=0}^{\lambda=l-2} \frac{1}{2^\lambda\,\lambda!}\ \log^\lambda\big(\sup(\frac{|\tau\,p|}{\kappa},\frac{\kappa}{m})\big)\ .$$

Assuming that $|p|>\kappa$ and also that $|p|\,m>\kappa^2$, which is the most delicate case (in the other cases some of the 3 contributions in (100) below are absent) we cut up the integral

$$\int_0^1 d\tau\ =\ \Big(\int_0^{\frac{\kappa}{p}}+\int_{\frac{\kappa}{p}}^{\frac{\kappa^2}{pm}}+\int_{\frac{\kappa^2}{pm}}^1\Big)d\tau$$

and find

$$|p|\Big(\int_0^{\frac{\kappa}{p}}+\int_{\frac{\kappa}{p}}^{\frac{\kappa^2}{pm}}+\int_{\frac{\kappa^2}{pm}}^1\Big)d\tau\frac{\log^\lambda\big(\sup(\frac{|\tau\,p|}{\kappa},\frac{\kappa}{m})\big)}{\sup(\tau|p|,\kappa)}\ \leq \log^\lambda(\frac{\kappa}{m})\ +\ \log^{\lambda+1}(\frac{\kappa}{m})\ +\ \frac{\log^{\lambda+1}(\frac{p}{\kappa})}{\lambda+1} \tag{100}$$

so that we obtain the bound

$$2\ \frac{K^{2l-1-\frac14}}{(l+1)^2}\ l!\ \sum_{\lambda=0}^{\lambda=l-2} \frac{1}{2^\lambda\,\lambda!}\ \log^\lambda(\frac{\kappa}{m}))\Big(1+\log(\sup(\frac{|p|}{\kappa},\frac{\kappa}{m}))\Big)\ .$$

The final lower bound on K is obtained on adding the bound (99) stemming from the boundary term at zero momentum and this one

$$2\,K^{-1/4} \,+\, K^{-5/4}\; K_2\; K_3\; 12\; \frac{(l+1)^2}{l^2} \;+\; 8\; K^{-1}(2\,K_0''\; K_1' \;+\; K_0''\; K_1) \;\leq\; 1\;. \tag{101}$$

In the second term we again replaced a factor of 6 by a factor of 2 as in (89).

b2) $|w| = 1$:

In this case we write

$$\partial\mathcal{L}_{2,l}(p) = \partial\mathcal{L}_{2,l}(0) \,+\, p\,\partial^2\mathcal{L}_{2,l}(0) \,+\, p^2 \int_0^1 d\tau\; (1-\tau)\; \partial^3\mathcal{L}_{2,l}(\tau p)\;. \tag{102}$$

Due to Euclidean symmetry the first term on the r.h.s. vanishes. The bound on the second term has been calculated in the previous section. The last term is bounded as in the previous calculation by

$$2\;\sup(p,\kappa)\; \frac{K^{2l-1-\frac{1}{4}}}{(l+1)^2}\; l! \sum_{\lambda=0}^{\lambda=l-2} \frac{1}{2^\lambda\,\lambda!}\; \log^\lambda(\frac{\kappa}{m}))\Big(1+\log(\sup(\frac{|p|}{\kappa},\frac{\kappa}{m}))\Big)\;.$$

so that we get again the lower bound (101) on K.

b3) $|w| = 0$:

We first look at $\mathcal{L}_{2,l}^{\Lambda,\Lambda_0}(0)$ which is written as

$$\mathcal{L}_{2,l}^{\Lambda,\Lambda_0}(0) \;=\; \mathcal{L}_{2,l}^{0,\Lambda_0}(0) \,+\, \int_0^\Lambda d\Lambda'\; \partial_{\Lambda'}\mathcal{L}_{2,l}^{\Lambda',\Lambda_0}(0)\;. \tag{103}$$

From the *first* term on the r.h.s. of the FE, where we use the bound (49) since two of the external momenta in $\mathcal{L}_{4,l-1}^{\Lambda',\Lambda_0}(0,0,k,-k)$ vanish, we obtain using again (31) and (57)

$$\begin{aligned}
&\binom{4}{2} \int_m^\kappa d\kappa' \int_k \frac{2}{\Lambda'^3}\; e^{-\frac{k^2+m^2}{\Lambda'^2}}\; \frac{K^{2l-2}}{l^2\,2^4}\;\; l! \sum_{\lambda=0}^{\lambda=l-1} \frac{\log^\lambda(\sup(\frac{|k|}{\kappa'},\frac{\kappa'}{m}))}{2^\lambda\,\lambda!} \\
\leq\;& \frac{6}{2^4}\; K_2\; K_3\; \frac{K^{2l-2}}{l^2}\; l! \sum_{\lambda=0}^{\lambda=l-1} \frac{1}{2^\lambda\,\lambda!} \int_m^\kappa d\kappa'\; \kappa'\; \Big(\log^\lambda(\frac{\kappa'}{m}) \,+\, (\lambda!)^{1/2}\Big) \\
\leq\;& \frac{6\,K_2\,K_3}{2^4}\; \frac{K^{2l-2}}{l^2}\; l! \sum_{\lambda=0}^{\lambda=l-1} \frac{1}{2^\lambda\,\lambda!}\; \frac{\kappa^2}{2}\; \Big(\log^\lambda(\frac{\kappa}{m}) \,+\, (\lambda!)^{1/2}\Big)
\end{aligned} \tag{104}$$

$$\leq\; 3\; \frac{6\,K_2\,K_3}{2^4}\; \frac{K^{2l-2}}{l^2}\; l! \sum_{\lambda=0}^{\lambda=l-1} \frac{1}{2^\lambda\,\lambda!}\; \frac{\kappa^2}{2}\; \log^\lambda(\frac{\kappa}{m})\;. \tag{105}$$

This is compatible with the induction hypothesis (52) if

$$K \geq \frac{9}{2^4} \, \frac{(l+1)^2}{l^2} \; K_2 \, K_3 \; . \tag{106}$$

Integrating the *second* term on the r.h.s. of the FE we obtain the bound

$$4 \int_0^\Lambda \frac{d\Lambda'}{\Lambda'^3} \; e^{-\frac{m^2}{\Lambda'^2}} \; \kappa'^4 \; K^{2l-2} \sum_{\substack{l_1+l_2=l, \\ l_1,l_2 \geq 1}} \frac{l_1!}{(l_1+1)^2} \, \frac{l_2!}{(l_2+1)^2} \sum_{\lambda_1=0}^{l_1-1} \frac{\log^{\lambda_1}(\frac{\kappa'}{m})}{2^{\lambda_1}\,\lambda_1!} \; \sum_{\lambda_2=0}^{l_2-1} \frac{\log^{\lambda_2}(\frac{\kappa'}{m}))}{2^{\lambda_2}\,\lambda_2!}$$

$$\leq \; 4 \; \frac{K^{2l-2}\,K_0''}{(l+1)^2} \; \int_0^\Lambda \frac{d\Lambda'}{\Lambda'^3} \; e^{-\frac{m^2}{\Lambda'^2}} \; \kappa'^2 \sum_{\lambda=0}^{\lambda=l-2} \frac{\log^{\lambda}(\frac{\kappa'}{m})}{2^{\lambda}\,\lambda!} \; \leq \; \kappa^2 \; \frac{K^{2l-1}}{(l+1)^2} \; l! \sum_{\lambda=1}^{\lambda=l-1} \frac{\log^{\lambda}(\frac{\kappa}{m})}{2^{\lambda}\,\lambda!}$$

using again Lemma 2c) and Lemma 7 and imposing the condition

$$4 \; K_0'' K_1 \; \leq \; K \; . \tag{107}$$

To go away from zero momentum we write similarly as in (102)

$$\mathcal{L}_{2,l}(p) = \; \mathcal{L}_{2,l}(0) \; + \; \frac{1}{2} \; p^2 \, \partial^2 \mathcal{L}_{2,l}(0) \; + \; p^3 \int_0^1 d\tau \; \frac{(1-\tau)^2}{2!} \; \partial^3 \mathcal{L}_{2,l}(\tau p) \tag{108}$$

and proceed in the same way as in the previous section, see (95), (96), (98), (100). Inductive verification of (51) gives similarly as in (101) the lower bound on K

$$K^{-\frac{1}{4}} + K^{-\frac{5}{4}} K_2 \; K_3 \; 6 \; \frac{(l+1)^2}{l^2} + K^{-1} \Big(\frac{1}{2} \, \frac{9}{2^4} \, \frac{(l+1)^2}{l^2} \; K_2 \, K_3 + 6 \; K_0'' K_1 + 8 \; K_0'' K_1' \Big) \leq 1 \tag{109}$$

noting that factors of $1/2$ are gained since

$$\sum_{\lambda=1}^{\lambda=l-1} \frac{\log^{\lambda}(\frac{\kappa}{m})}{2^{\lambda}\,\lambda!} \; \leq \; \frac{1}{2} \; \sum_{\lambda=0}^{\lambda=l-2} \frac{\log^{\lambda}(\frac{\kappa}{m})}{2^{\lambda}\,\lambda!} \; (1 \; + \; \log^{\lambda}(\frac{\kappa}{m})) \; .$$

■

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